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REGULAR AND SEMISIMPLE MODULES

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A module is regular if all its submodules are (Cohn) pure. The family of all regular modules is closed under products if and only if $R/J(R)$ is a von Neumann regular ring. If each regular R-module is semisimple then R is a T-ring. An extra condition is needed for the converse. Character modules and extensions of regular and semisimple modules are investigated.

1. Introduction. Rings will be associative with identity and modules will be (left) unitary. R will denote a ring which is not assumed commutative unless specifically stated and $J(R)$ will denote the Jacobson radical of R. Fieldhouse [5] calls a module B regular if each submodule A of B is pure in B, i.e., the inclusion $0 \to A \to B$ remains exact upon tensoring by any (right) R-module. Regular modules have been studied under different definitions by Ware [12], Zelmanowitz [14], and Ramamurthi and Rangaswamy [9]. A module is semisimple if it is a sum of simple modules. For a subset A of a module B, $(0: A)$ will denote the left ideal $\{r \in R \mid rx = 0$ for all $x \in A\}$.

2. Products. The class of all semisimple modules is closed under products if and only if $R/J(R)$ is a semisimple (Artinian) ring. This follows from the canonical embedding $R/J(R) \hookrightarrow \Pi R/M$, where the product is taken over the set of maximal left ideals M of R.

LEMMA 1. *If I is a two-sided ideal of R, then R/I is a regular ring if and only if R/I is a regular left* (or right) *R-module.*

Proof. For any left R/I-module B and any right R-module C we have canonical group isomorphisms:

$$C \otimes_R B \simeq (C/CI) \otimes_R B \simeq (C/CI) \otimes_{R/I} B .$$

If follows that an R/I-module is regular as an R/I-module if and and only if it is regular as an R-module. This proves the lemma.

COROLLARY. *If R is a commutative ring, an R-module B is regular if and only if $R/(0: x)$ is a regular ring for each $0 \neq x \in B$.*

THEOREM 1. *The following statements are equivalent for a ring R.*