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CONVEX AND CONCAVE FUNCTIONS OF SINGULAR VALUES OF MATRIX SUMS

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A new set of inequalities for functions of singular values of matrix sums is established. These inequalities are complementary to a number of classically known inequalities in that the direction of the inequality sign is reversed. A matrix valued triangle inequality is also given. Special cases of these results are due to S. Yu. Rotfel'd.

1. Introduction. Let A, B, $C = A + B$ be n-square matrices with singular values $\alpha_1 \geqq \cdots \geqq \alpha_n$, $\beta_1 \geqq \cdots \geqq \beta_n$, $\gamma_1 \geqq \cdots \geqq \gamma_n$ respectively. In **[5]** S. Yu. Rotfel'd stated and in **[6]** proved that

$$f(\gamma_1) + \cdots + f(\gamma_n) \leqq f(\alpha_1) + \cdots + f(\alpha_n) + f(\beta_1) + \cdots + f(\beta_n) \tag{1}$$

when f is an increasing concave function of a nonnegative real variable, with $f(0) = 0$. This inequality is of some interest as in previously published work a convexity (rather than concavity) hypothesis has usually been necessary to establish results of the general type of (1). See, for example, Gohberg and Krein **[3]**, page 49, or Marcus and Minc **[4]**, pages 103 and 116. In this paper we shall uncover the algebraic foundation of (1) by giving a short proof of a generalization. Our proof, which is simpler and more direct than the proof of (1) given by Rotfel'd, will be based on an interesting matrix valued triangle inequality, a special case of which was given by Rotfel'd. We note that the methods used by Rotfel'd are very much adapted to the inequality (1) that he wished to prove, and do not appear capable of proving the extensions of his results to be presented below.

2. Positive semidefinite matrices. In this section we discuss an important special case for which sharper results are possible. Throughout §2 we let A, B, $C = A + B$ be $n \times n$ Hermitian positive semidefinite matrices with eigenvalues $\alpha_1 \geqq \cdots \geqq \alpha_n$, $\beta_1 \geqq \cdots \geqq \beta_n$, $\gamma_1 \geqq \cdots \geqq \gamma_n$, respectively. Let $F(x_1, \cdots, x_{2n})$ and $G(x_1, \cdots, x_{2n})$ be functions of $2n$ nonnegative variables, symmetric in these variables (i.e., remaining unchanged if the variables are permuted), with F concave and G convex: for all nonnegative vectors x, y and real numbers θ with $0 \leqq \theta \leqq 1$,

$$F(\theta x + (1-\theta)y) \geqq \theta F(x) + (1-\theta)F(y),$$

$$G(\theta x + (1-\theta)y) \leqq \theta G(x) + (1-\theta)G(y).$$