PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No 2, 1976

RINGS WHOSE ADDITIVE SUBGROUPS ARE SUBRINGS

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In this paper the class (subclass) of associative rings whose additive subgroups are subrings (ideals) is completely characterized by defining relations. An exact description is also given of those rings in these classes which are commutative, regular, Artinian, Noetherian, or with identity. The only integral domains in either class are the ring of integers Z and $Z/(p)$ for prime p.

0. Introduction. A ring which has the property, called S, that all its additive subgroups are subrings is called a S-ring. A ring R is a S-ring if and only if, for every x and y in R, xy is a linear combination of x and y. This fact will be used constantly in this paper. All rings herein are associative, all groups are abelian, and all constants are integers. A ring R will be primary, torsion, torsion-free, mixed, etc. exactly if its additive group, R^+, is primary, etc. By the rank of R, $r(R)$, we mean the rank of R^+. Occasionally, when the meaning is perfectly clear, distinction will not be made between R and R^+. Other terminology is essentially that of [4]. Our chief results are found in Theorems 1.4, 1.5, 1.6, 2.6, 3.5, and 3.6. Special S-rings are described in §4.

1. Torsion rings. In this section all rings are torsion. We begin with primary rings.

PROPOSITION 1.1. *If R is a p-primary S-ring, R is bounded or R is a zero-ring.*

Proof. (a) Let B be a basic subgroup of R^+ and assume B is unbounded. We prove $R^2 = 0$. Suppose $x^2 = rx \neq 0$ for some x in B and $o(x) = p^n$. Choose z independent of x such that $o(z) \geqq p^{2n}$. Then $(x + p^n z)^2 = rx + p^{2n} z = k(x + p^n z)$ for some k. This implies p^n divides k and $0 = kx = rx$, a contradiction. Thus $x^2 = 0$, for every x in B. Suppose x and y are linearly independent in B and $xy = ax + by$, $by \neq 0$. If $o(x) = p^n$ and $o(y) = p^m$, choose z in B such that z is independent of x and y and p^{n+m} divides $o(z)$. Write $xz = cx + dz$. Then $x(y + z) = (a + c)x + (by + dz)$. The right-hand term must be a multiple of $y + z$ which implies $dy = by \neq 0$. But $p^n dz = p^n(xz) = 0$ and p^m divides d which implies $dy = 0$. Therefore, $xy = 0$ and $B^2 = 0$. It follows easily that $R^2 = 0$ (see Theorem 120.1 of