PACIFIC JOURNAL OF MATHEMATICS
Vol. 66, No. 2, 1976

LEVEL SETS OF POLYNOMIALS IN n REAL VARIABLES

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The methods used in studying the zeros of a polynomial in a single complex variable are here adapted to investigating the level surfaces of a real polynomial in E^n, with respect to their intersection and finite or asymptotic tangency with certain cones. Special attention is given to the equipotential surfaces generated by an axisymmetric harmonic polynomial in E^3.

A principal interest is the application of reasoning used by Cauchy [**2**, p. 123] in obtaining bounds on the zeros of polynomials in one complex variable. We thereby seek the level sets

$$L_\alpha(H) = \{X \in E^n \mid H(X) = \alpha\}$$

generated from the real polynomials

$$H(X) - \alpha = \sum_{0 \leqq j_1 + \cdots + j_n \leqq n} \alpha_{j_1 \cdots j_n} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}, \tag{1}$$

$$X = (x_1, x_2, \cdots, x_n),\ r = |X| = [x_1^2 + x_2^2 + \cdots + x_n^2]^{1/2}.$$

It is convenient to introduce direction numbers $\lambda_j = x_j r^{-1}$, $1 \leqq j \leqq n$, connected by $\lambda_1^2 + \cdots + \lambda_n^2 = 1$ and cones $\Lambda_j : \lambda_j =$ constant, about the jth axis. On the intersection of the cones Λ_j, these polynomials become

$$H(r\Lambda_j) - \alpha = \sum_{k=0}^{n} r^k A_k(\Lambda_j)$$

where

$$A_k = A_k(\Lambda_j) = \sum_{j_1 + \cdots j_n = k} \alpha_{j_1 \cdots j_n} \lambda_1^{j_1} \cdots \lambda_n^{j_n}, \qquad 0 \leqq k \leqq n.$$

At the origin the level set $L_\alpha(H)$ has νth order contact with Λ_j, if $A_k(\Lambda_j) = 0$ for $0 \leqq k \leqq \nu - 1$ but $A_\nu(\Lambda_j) \neq 0$ and $A_n(\Lambda_j) \neq 0$. For such sets we introduce the ratios

$$M_\nu = M_\nu(\Lambda_j) = \max_{\nu \leqq k \leqq n-1} |A_k / A_n|$$