SOME REMARKS ON CONVOLUTION EQUATIONS
FOR VECTOR-VALUED DISTRIBUTIONS

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Let $E$, $X$ be two complex Banach spaces, $(E; X)$ the space of all linear bounded operators from $E$ into $X$ endowed with its usual norm. We denote by $\mathcal{D}'(E)$ the space of distributions with values in $E$ defined in $-\infty < t < \infty$ and by $\mathcal{D}_c'(E)$ the subspace of $\mathcal{D}'(E)$ consisting of all $T \in \mathcal{D}'(E)$ with support in $t \geq 0$.

Given $P \in \mathcal{D}_c'(E; X)$ we examine the following problems.

(I). Does $P$ have a convolution inverse with support in $t \geq 0$, that is, is there $S \in \mathcal{D}_c'(E; X)$ satisfying

\[
P * S = \delta \otimes I, \quad S * P = \delta \otimes J
\]

where $I$ (resp. $J$) denotes the identity operator in $E$ (resp. $X$) and $\delta$ is the Dirac delta?

(II). In case the answer to (I) is affirmative, what properties of $S$ can be deduced from properties of $P$ and vice versa?

These problems have been exhaustively studied in the case where $P = \delta' \otimes I - \delta \otimes A$, $A$ a closed, densely defined operator in $E$ and $X = D(A)$ with its graph norm. When $S$ is strongly continuous in $t \geq 0$ then $S$ is a strongly continuous semigroup and $A$ its generator, characterized by the Hille–Yosida–Phillips theorem ([6], Ch. VIII); in the general case ($S \in \mathcal{D}_c'(E; X)$) $S$ is a distribution semigroup in the sense of Lions [15] and its generator $A$ is characterized by a result of Chazarain ([2], [3]; see also [19]). Several subcases and variants have been studied by Pazy [20], Barbu [1], Da Prato–Mosco [4], [5], Foias [12], Fujiwara [13], Yosida [25] and others (see [15] for the semigroup case). Similar problems for more general differential operators have also been considered; see for instance [8], [9], [10], [11], [24].

Our aim is to show here that many of the results just mentioned extend to the general case (with no restrictions on $P$, except perhaps on the support of $P$ or on its growth at infinity) although some interesting new phenomena appear. Motivation for this extension is provided by the fact that many state equations arising in applications are not purely differential; for instance equations describing the behavior of materials with memory (which appear in magnetic hysteresis, viscoelasticity, etc). We examine in detail in §8 a heat equation proposed by Gurtin and Pipkin [14] where the temperature at a given time depends on the temperature history of the system.