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A GENERALIZATION OF THE UNIT INTERVAL

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Convex sets are discussed here in linear spaces over scalars other than the reals. To facilitate this development, the interval $[0,1]$ is generalized to a unit interval in an arbitrary division ring. The interval $[0,1]$ is shown to be the maximum generalized unit interval in the real number field, the complex number field, and the quaternion division ring. Several elementary theorems on convexity are proved for linear spaces over scalars having generalized unit intervals of certain types.

For a real vector space $\mathcal{V}$, a set $A \subset \mathcal{V}$ is *convex* iff $\forall x, y \in A$, $\forall \lambda \in [0,1]$ $(1-\lambda)x + \lambda y \in A$. As might be expected, some of the vector space axioms are not needed when study is restricted to the convex subset itself. In fact, instead of the two maps:

$$\begin{aligned} D\colon \mathcal{R} \times \mathcal{V} &\to \mathcal{V} \\ (a, x) &\to ax \\ E\colon \mathcal{V} \times \mathcal{V} &\to \mathcal{V} \\ (x, y) &\to x + y \end{aligned}$$

satisfying the usual vector space axioms, all that is required is a map:

$$\begin{aligned} T\colon [0,1] \times A \times A &\to A \\ (\lambda, x, y) &\to (1-\lambda)x + \lambda y \end{aligned}$$

such that $\forall \lambda, \mu \in [0,1]$ and $\forall x, y, z \in A$

(1) $(1-\lambda)x + \lambda y = \lambda y + (1-\lambda)x$

(2) $(1-\lambda)x + \lambda[(1-\mu)y + \mu z]$
$$= (1-\lambda\mu)\left[\left(1 - \frac{(1-\lambda)\mu}{1-\lambda\mu}\right)x + \frac{(1-\lambda)\mu}{1-\lambda\mu}y\right] + \lambda\mu z$$

(3) $(1-\lambda)x + \lambda x = x$

(4) $(1-\lambda)x + \lambda y = (1-\lambda)x + \lambda z$ implies $y = z$.

These axioms have been studied in some depth **[1, 6]** and have been applied to axiomatic quantum theory **[1, 2, 6]**. Such a development of convex structures lacks much of the generality that is possible. For example, the cardinality of points on a line segment is restricted to the cardinality of the continuum. Also, the scalars used in axiomatic quantum theory should be the complex numbers **[3]**, or possibly the