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RATIONAL APPROXIMATION TO x^n

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This note is concerned with the approximations of x^n on $[0,1]$ by polynomials and rational functions having only non-negative coefficients and of degree at most $k (1 \leqq k \leqq n-1)$. It is shown that the best approximating polynomial of degree k on $[0,1]$ to x^n is of the form

$$p_k(x) = dx^k ,$$

where $d > 0$ and satisfies the assumption that

$$n(1-d) = (n-k)\left(\frac{k}{n}\right)^{k/(n-k)} d^{n/(n-k)} ,$$

with an error $\varepsilon_k = 1 - d$, for each fixed $k = 1, 2, 3, \cdots, n-1$. It is also shown that dx^k is a best approximating rational function of degree k to x^n on $[0,1]$.

More than one hundred years ago Chebyshev showed that x^n can be uniformly approximated on $[-1,1]$ by polynomials of degree at most $(n-1)$ with an error of exactly 2^{-n+1}.

Just recently D. J. Newman [1] has shown that x^n can be uniformly approximated on $[-1,1]$ by rational functions of degree at most $(n-1)$ with an error roughly $\sqrt{n}(3\sqrt{3})^{-n}$.

If one looks carefully at the above results, then the following questions arise naturally.

Q.1: How close can one approximate x^n uniformly on $[0,1]$ by polynomials of degree $(n-1)$ having only non-negative coefficients?

Q.2: Is the error obtained by rational functions of degree $(n-1)$ having only nonnegative coefficients in approximating x^n on $[0,1]$ less than the error obtained by polynomials of degree $(n-1)$ having only nonnegative coefficients?

We answer these questions in this note.

Let

$$(1) \qquad \varepsilon_k = \inf_{p \in \pi_k^*} ||x^n - p(x)||_{L\infty[0,1]}$$

where $\pi_k^* (1 \leqq k < n)$ denotes the class of all algebraic polynomials of degree at most k having only nonnegative coefficients.

$$(1') \qquad \theta_k = \inf_{p,q \in \pi_k^*} \left|\left| x^n - \frac{p(x)}{q(x)} \right|\right|_{L\infty[0,1]} .$$