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AN UPPER BOUND FOR THE PERIOD OF THE SIMPLE CONTINUED FRACTION FOR $\sqrt{D}$

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Let $p(D)$ denote the length of the period of the simple continued-fraction expansion of $\sqrt{D}$, where D is a positive non-square integer. In this paper, it is shown that

$$p(D) < 0.72D^{1/2} \log D$$

for all squarefree $D > 7$, and an estimate for $p(D)$ is given when D is not squarefree.

1. Introduction. The problem of finding a good upper bound for the length $p(D)$ of the period of the simple continued fraction for $\sqrt{D}$, where D is a positive nonsquare integer, has received relatively little attention. Recently, Hickerson [6] and Hirst [7] have given estimates for $p(D)$; Hickerson's estimate implies that

$$\log p(D) < \log D(1/2 + \log 2(\log \log D)^{-1} + o(\log \log D)^{-1}) , \tag{1.1}$$

where D is nonsquare, and Hirst's implies that

$$p(D) < 2D^{1/2} \log D + 0(D^{1/2}) , \tag{1.2}$$

where D is squarefree. Both authors give more precise error terms, but these are not relevant here. For general nonsquare $D > 0$, Hirst shows that

$$p(D) = O(D^{1/2} s \log D) \tag{1.3}$$

uniformly in s, where s^2 is the largest square factor of D. For sufficiently large squarefree D, (1.2) is clearly better than (1.1). On the other hand, (1.3) is better than (1.1) only when s, regarded as a function of D, is sufficiently small. Pen and Skubenko [14] have given an upper bound for $p(D)$ which we will discuss later; it depends on the size of the least positive solution of $x^2 - Dy^2 = 1$.

The authors [17] have used combinatorial methods to show that

$$p(D) < 0.82D^{1/2} \log D$$

for all squarefree $D > 7$. In this paper, we use a different approach which refines this result to

$$p(D) < 0.72D^{1/2} \log D \tag{1.4}$$

for all squarefree $D > 7$. It is also shown that

$$p(D) < 3.76D^{1/2} \log (D/s^2) \tag{1.5}$$