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THE ALTITUDE FORMULA AND DVR's

L. J. RATLIFF, JR.

The main theorem in this paper characterizes a local domain (R, M) which satisfies the altitude formula in terms of certain DVR's (discrete valuation rings) in the quotient field F of R. Specifically, R satisfies the altitude formula if and only if every DVRPL (V, N) over R in F (that is, (V, N) is a DVR with quotient field F such that $R \subseteqq V$, $N \cap R = M$, and V is integral over a locality over R) is of the first kind (that is, trd $(V/N)/(R/M) =$ altitude $R - 1$).

Such a characterization is of interest and importance, since it is related to the Chain Conjecture 2.18.1. Thus, by 2.19.1, the Chain Conjecture holds if and only if every DVRPL over (I, P) in F is of the first kind, where $I = R'_{M'}$ with M' a maximal ideal in the integral closure R' of a local domain R.

The theorem is also related to the following well known result [1, Proposition 4.4] or [14, Proposition 5.1]: If (R, M) is a regular local ring and (V, N) is a valuation ring in the quotient field F of R such that $R \subseteqq V$, $N \cap R = M$, and trd $(V/N)/(R/M) =$ altitude $R - 1$, then V is a DVR and the sequence of quadratic transformations of R along V is finite. In 2.22.1 it is shown that the same conclusion holds when R is an analytically unramified local domain which satisfies the altitude formula. Moreover, 2.11 implies that a form of the converse also holds, namely: If (W, Q) is a valuation ring in F such that $R \subseteqq W$, $Q \cap R = M$ and the sequence of quadratic transformations of R along W is finite, then W is a DVR, W is a locality over R, and trd$(V/N)/(R/M) =$ altitude $R - 1$ 2.22.2. A number of other corollaries to 2.11 are given in 2.12–2.22.

In § 3 we consider a closely related subject. Namely, we prove some results concerning a locality (S, P) over a local domain (R, M) such that $P \cap R = M$ and S contains elements with property (T) 3.1. For such S, S satisfies the altitude formula 3.2.1. Also, every DVRPL V^* over S in E is such that $V^* \cap F$ is a DVRPL over R of the first kind, where E and F are the quotient fields of S and R, respectively 3.2.2. Further, either R satisfies the altitude formula or, for each DVRPL V over R in F which is of the second kind, $PV[S] = V[S]$ 3.2.3. Moreover, for each maximal ideal P' in the integral closure S' of S, $S'_{P'} \cap F$ is a DVRPL over R of the first kind 3.3. It is then shown that if R is analytically unramified and E is separably generated over R, then, for each maximal ideal P' in S', $S'_{P'}$ is analytically irreducible 3.4. Two further related corollaries are also