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ON EXTREME POINTS OF THE JOINT NUMERICAL RANGE OF COMMUTING NORMAL OPERATORS

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Let $W(T) = \{\langle Tx, x\rangle : \|x\| = 1; x \in H\}$ denote the numerical range of a bounded normal operator T on a complex Hilbert space H. S. Hildebrandt has proved that if λ is an extreme point of $\overline{W(T)}$, the closure of $W(T)$, and $\lambda \in W(T)$ then λ is in the point spectrum of T. In this note, we shall prove an analogous result for an n-tuple of commuting bounded normal operators on H.

2. Notations and terminology. Let $A = (A_1, \cdots, A_n)$ be an n-tuple of commuting bounded operators on H and $\mathscr{U}$, the double commutant of $\{A_1, \cdots, A_n\}$. Then $\mathscr{U}$ is a commutative Banach algebra with identity, containing the set $\{A_1, \cdots, A_n\}$. We shall need the following definitions [3] and [4].

A point $z = (z_1, \cdots, z_n)$ of $\mathscr{C}^n$ is in the joint spectrum $\sigma(A)$ of A relative to $\mathscr{U}$ if for all $B_1, \cdots, B_n$ in $\mathscr{U}$

$$\sum_{j=1}^{n} B_j(A_j - z_j) \neq I .$$

The joint numerical range of A is the set of all points $z = (z_1, \cdots, z_n)$ of $\mathscr{C}^n$ such that for some x in H with $\|x\| = 1$, $z_j = \langle A_j x, x\rangle$ i.e.,

$$W(A) = \{\langle Ax, x\rangle = (\langle A_1 x, x\rangle, \cdots, \langle A_n x, x\rangle)\} .$$

We say that $z = (z_1, \cdots, z_n)$ is in the joint point spectrum $\sigma_p(A)$ if there exists some $0 \neq x \in H$ such that

$$A_j x = z_j x , \qquad j = 1, \cdots, n ,$$

and that z is in the joint approximate point spectrum $\sigma_\pi(A)$ if there exists a sequence $\{x_n\}$ of unit vectors in H such that $\|(A_j - z_j)x_n\| \to 0$ as $n \to \infty$, $j = 1, \cdots, n$.

Bunce [2] has proved that $\sigma_\pi(A)$ is a nonempty compact subset of $\mathscr{C}^n$.

If $A = (A_1, \cdots, A_n)$ is an n-tuple of commuting normal operators, then the extreme points of $\overline{W(A)}$ are in the joint approximate point spectrum $\sigma_\pi(A)$. This is immediate from the fact that for such A_j's,

$$\begin{aligned}\overline{W(A)} &= \text{closed convex hull of } \sigma(A)\\ &= \text{closed convex hull of } \sigma_\pi(A) ,\end{aligned}$$

and that every compact set contains the extreme points of its closed