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CHARACTERIZATIONS OF CERTAIN MAPS OF CONTRACTIVE TYPE

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The following result is obtained. Let f be a self map on a nonempty complete metric space (X, d). Then the following conditions are equivalent: (i) For any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$. (ii) There exists a function w of $[0, \infty)$ into $[0, \infty)$ such that $w(s) > s$ for all $s > 0$, w is lower semicontinuous from the right on $(0, \infty)$ and $w(d(f(x), f(y))) \leqq d(x, y)$, $x, y \in X$.

1. Introduction. In 1969, E. Keeler and A. Meir [3] obtained the following result.

THEOREM A. (Keeler and Meir). *Let f be a self map on a nonempty complete metric space (X, d). Suppose that for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $\epsilon \leqq d(x, y) < \epsilon + \delta(\epsilon)$. Then f has a unique fixed point x_0 and $\{f^n(x)\}$ converges to x_0 for all x in X.*

Theorem A generalized the following result of D. W. Boyd and J. S. W. Wong [1] (and therefore, an earlier result of E. Rakotch [4]).

THEOREM B. (Boyd and Wong). *Let f be a self map on a nonempty complete metric space (X, d). Suppose that there exists a self map Φ on $[0, \infty)$ such that Φ is upper semicontinuous from the right, $\Phi(t) < t$ for $t > 0$ and f is Φ-contractive:*

$$d(f(x), f(y)) \leqq \Phi(d(x, y)), \qquad x, y \in X.$$

Then f has a unique fixed point x_0 and $\{f^n(x)\}$ converges to x_0 for all x in X.

In this paper, equivalent conditions in terms of monotone transformations are obtained. These will show that the essential difference between Theorems A and B is a matter of imposing monotone transformations on the left side or right side of certain inequalities.

2. Main results.

THEOREM 1. *Let f be a self map on a nonempty complete metric space (X, d). Then the following conditions are equivalent:*