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ON A THEOREM OF APOSTOL CONCERNING MÖBIUS FUNCTIONS OF ORDER k

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In 1970, Tom M. Apostol introduced a class of arithmetical functions $\mu_k(n)$ for all positive integral k, as a generalization of the Möbius function $\mu(n) = \mu_1(n)$ and established the following theorem: For $k \geqq 2$, $M_k(x) = \Sigma_{n \leqq x}\mu_k(n) = A_k x + O(x^{1/k} \log x)$, where A_k is a positive constant. In this paper we improve the above O-estimate to $O(x^{4k/(4k^2+1)}\omega(x))$ on the assumption of the Riemann hypothesis, where $\omega(x) = \exp\{A \log x (\log \log x)^{-1}\}$, A being a positive absolute constant.

1. Introduction. T. M. Apostol [1] introduced the following generalization of the Möbius function $\mu(n)$. Let k be a fixed positive integer. Let μ_k, the Möbius function of order k be defined by $\mu_k(1) = 1$, $\mu_k(n) = 0$ if $p^{k+1} | n$ for some prime p, $\mu_k(n) = (-1)^r$ if $n = p_1^k p_2^k \cdots p_r^k \Pi_{i>r} p_i^{a_i}$, $0 \leqq a_i < k$, $\mu_k(n) = 1$ otherwise. In other words, $\mu_k(n)$ vanishes if n is divisible by the $(k+1)$st power of some prime; otherwise, $\mu_k(n)$ is 1 unless the prime factorization of n contains the kth powers of exactly r distinct primes, in which case $\mu_k(n) = (-1)^r$. When $k = 1$, $\mu_k(n)$ is the usual Möbius function, $\mu_1(n) = \mu(n)$.

He established the following asymptotic formula (cf. [1], Theorem 1) for the summatory function $M_k(x) = \Sigma_{n \leqq x}\mu_k(n)$: For $k \geqq 2$ and $x \geqq 2$

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O(x^{1/k} \log x), \tag{1}$$

where A_k is the constant given by

$$A_k = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{k+1}}\right), \tag{2}$$

the product being extended over all primes p.

In this note we improve the O-estimate of the error term in (1) above on the assumption of the Riemann hypothesis by proving the following: For $x \geqq 3$,

$$\sum_{n \leqq x} \mu_k(n) = A_k x + O\,(x^{4k/(4k^2+1)}\omega(x)), \tag{3}$$

where $\omega(x)$ is given by (5) below.