PACIFIC JOURNAL OF MATHEMATICS
Vol. 68, No. 1, 1977

STIEFEL–WHITNEY CLASSES OF MANIFOLDS

R. E. STONG

1. Introduction. The purpose of this paper is to axiomatize the Stiefel–Whitney classes of closed manifolds and using the axioms to give a proof of Wu's theorem ([3]):

THEOREM. *If M^n is a closed n manifold and $v = 1 + v_1 + \cdots + v_n$ with $v_i \in H^i(M; Z_2)$ defined by*

$$\langle v_i \cup x, [M]\rangle = \langle Sq^i x, [M]\rangle$$

for all $x \in H^{n-i}(M; Z_2)$, then $w = Sqv$ is the Stiefel–Whitney class of M.

The axioms for the Stiefel–Whitney classes are as follows.

For each closed manifold M^n there is a class $\alpha(M) = \alpha_0(M) + \cdots + \alpha_n(M)$ where $\alpha_i(M) \in H^i(M; Z_2)$ satisfying:

(1) If $i: M \to N$ is an imbedding with trivial normal bundle, then $\alpha(M) = i^*\alpha(N)$,

(2) $\alpha(M \times N) = \pi_M^*(\alpha(M)) \cup \pi_N^*(\alpha(N))$, and

(3) $\alpha(RP(n)) = (1 + a)^{n+1}$, where $a \in H^1(RP(n); Z_2)$ is the nonzero class.

It is well-known (See [2]) that the Stiefel–Whitney classes satisfy these properties, and it will be shown that if $\alpha(M)$ satisfies these properties, then $\alpha(M) = w(M)$.

A different axiomatization has been given by J. D. Blanton and P. A. Schweitzer, "Axioms for characteristic classes of manifolds", Proc. of Symp. in Pure Math., Amer. Math. Soc. 27 (1975), volume I, 349–356. Their axioms use all manifolds, so cannot be used to prove Wu's theorem which is only meaningful for closed manifolds.

2. Axiomatics. Suppose one is given classes $\alpha(M) \in H^*(M; Z_2)$ for each closed manifold M^n which satisfy properties 1, 2, and 3.

LEMMA 1. $\alpha_0(M) = 1$, *i.e., is the unit class.*

Proof. If P is a point and $i: P \to RP(n)$ then i is an imbedding with trivial normal bundle so $\alpha_0(P) = 1$. Then for any $f: P \to M$, f is an imbedding with trivial normal bundle, so $f^*\alpha_0(M) = 1$. Thus $\alpha_0(M) = 1$.*