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ANTI-COMMUTATIVE ALGEBRAS AND HOMOGENEOUS SPACES WITH MULTIPLICATIONS

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As a generalization of certain results for Lie groups it is shown that an n-dimensional H-space (M, μ) with identity e has a coordinate system at e in which μ can be represented by a function $F: R^n \times R^n \to R^n$ which is analytic at $(0,0)$ and that the second derivative of F induces a bilinear anti-commutative multiplication α on R^n. In this way an algebra (R^n, α) analogous to the Lie algebra of a Lie group is obtained and all such algebras are shown to be isomorphic. If $M = G/H$ is a reductive homogeneous space, then these results generalize the Lie group–Lie algebra correspondence and the algebra (R^n, α) induces a G-invariant connection on G/H. Relative to this connection it is shown that an automorphism of $(G/H, \mu)$ is an affine map and induces an algebra automorphism of (R^n, α). Also the connection is irreducible if $(G/H, \mu)$ has no proper invariant subsystems (the analog of normal subgroups). In the case where G/H has a Riemannian structure, it may happen that there are no local isometries among the coordinate maps which give rise to anti-commutative multiplications on R^n.

1. Multiplications and change of coordinates. Let M be an n-dimensional real, analytic manifold and let $\mu: M \times M \to M$ be an analytic function such that $\mu(e, e) = e$ for some $e \in M$. In this case μ is called a *multiplication* on M and we denote this *multiplicative structure* by (M, μ). In the examples we consider, e is a two-sided identity element; that is, (M, μ) is an H-space (for other examples see [6]). In particular we will consider Lie groups and Moufang loops [1, 8].

For the multiplicative structure (M, μ) let (U, ϕ) be a coordinate system at $e \in M$ where U is a neighborhood of e and $\phi: U \to R^n$ is the coordinate map. Assume that $\phi(e) = 0$ in R^n and let $\phi^{-1}: U_0 \to M$ denote the local inverse function of ϕ defined on a neighborhood U_0 of 0. For $D \subset U_0$ a suitable neighborhood of $0 \in R^n$ we can represent μ in the coordinate system $(\phi^{-1}(D), \phi|_{\phi^{-1}(D)})$ as $\mu(\phi^{-1}X, \phi^{-1}Y) = \phi^{-1}F(X, Y)$ for $X, Y \in D$ where $F: D \times D \to U_0$ is analytic at $(0,0) \in D \times D$ and defines a "local multiplicative structure" (U_0, F).

Let $\theta = (0,0)$; then since F is analytic we can form the kth derivative $F^k = F^k(\theta)$, which is a symmetric k-multilinear form on R^n and, using the notation $F^kZ^{(k)} = F^k(Z, Z, \cdots, Z)$, with $Z = (X, Y)$, we can write