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NORMS OF COMPACT PERTURBATIONS OF OPERATORS

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Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex separable Hilbert space. This paper is concerned with reducing the norm of a product of operators by compact perturbations of one or more of the factors. For any T in $\mathscr{B}(\mathscr{H})$, it is well known that the infimum,

$$\|T\|_e = \inf\{\|T+K\|: K \text{ is a compact operator}\}$$

is attained by some compact perturbation $T+K_0$. For T a noncompact product of n operators, $T = T_1 \cdots T_n$, it is proved that this infimum can be obtained by a compact perturbation of any one of the factors. If T is a compact product, so that the infimum is zero, it is shown that there are compact perturbations $T_1 + K_1, \cdots, T_n + K_n$ of the factors of T such that the product $(T_1 + K_1)\cdots(T_n + K_n)$ is zero; furthermore, it may be necessary to perturb every factor of T in order to obtain this zero infimum. These results are applied to an arbitrary operator T to find a compact perturbation $T+K$ with $\|(T+K)^2\| = \|T^2\|_e$ and $\|(T+K)^3\| = \|T^3\|_e$; here the identical factors are perturbed in identical fashion to achieve both infima. Stronger theorems of this latter sort are proved for special classes of operators.

For any T in $\mathscr{B}(\mathscr{H})$, let $\|T\|_e$ as defined above, be called the *essential norm* of T **[7]**. I. C. Gohberg and M. G. Krein first showed in **[4]** that for any T in $\mathscr{B}(\mathscr{H})$ there is a compact perturbation $T + K_0$ which realizes the essential norm (so $\|T+K_0\| = \|T\|_e$). The case $n = 2$ of the theorem stated above for compact products was proved in a different way in **[6]**: for any compact product $T = T_1T_2$ of two factors, a projection E was constructed so that T_1E and $(I-E)T_2$ are both compact (and so that the product of perturbations $T_1(I-E)$ and ET_2 is zero).

This study was motivated partly by questions considered by J. K. Plastiras and the author in **[7]**: if T is a bounded operator on $\mathscr{H}$, is there a compact K with $\|p(T+K)\| = \|p(T)\|_e$ for all complex polynomials p? Less ambitiously, if T and p are both given, is there a compact K_p such that $\|p(T+K_p)\| = \|p(T)\|_e$? We know of no examples where either of these questions has a negative answer.

It follows from the results proved here on perturbations of products that for each T in $\mathscr{B}(\mathscr{H})$, there is a compact K with $\|T+K\| = \|T\|_e$ and $\|(T+K)^2\| = \|T^2\|_e$; and a compact L with $\|(T+L)^2\| = \|T^2\|_e$ and $\|(T+L)^3\| = \|T^3\|_e$. If T^3 is not compact we can take $K = L$, to get one