PACIFIC JOURNAL OF MATHEMATICS
Vol. 68, No 1, 1977

SUPERALGEBRAS OF WEAK-*DIRICHLET ALGEBRAS

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Let A be a weak-*Dirichlet algebra of $L^\infty(m)$ and let $H^\infty(m)$ denote the weak-*closure of A in $L^\infty(m)$. Muhly showed that if $H^\infty(m)$ is an integral domain, then $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$. We show in this paper that if $H^\infty(m)$ is not maximal as a weak-*closed subalgebra of $L^\infty(m)$, there is no algebra which contains $H^\infty(m)$ and is maximal among the proper weak-*closed subalgebras of $L^\infty(m)$. Moreover, we investigate the weak-*closed superalgebras of A and we try to classify them. We show that there are two canonical weak-*closed superalgebras of A which play an important role in the problem of describing all the weak-*closed superalgebras of A.

1. Preliminaries. Recall that by definition **[7]**, a weak-*Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathscr{A}, m)$ such that (i) the constant functions lie in A; (ii) $A + \bar{A}$ is weak-*dense in $L^\infty(m)$ (the bar denotes conjugation, here and always); (iii) for all f and g in A, $\int_X fgdm = \int_X fdm \int_X gdm$. The abstract Hardy space $H^p(m)$, $1 \leqq p \leqq \infty$, associated with A are defined as follows. For $1 \leqq p \leqq \infty$, $H^p(m)$ is the $L^p(m)$-closure of A, while $H^\infty(m)$ is defined to be the weak-*closure of A in $L^\infty(m)$. For $1 \leqq p \leqq \infty$, let $H_0^p = \left\{f \in H^p(m); \int_X fdm = 0\right\}$.

A (weak-*closed) subalgebra B^∞ of $L^\infty(m)$, containing A, is called a superalgebra of A. Let $B_0^\infty = \left\{f \in B^\infty; \int_X fdm = 0\right\}$ and let I_B^∞ be the largest weak-*closed ideal of B^∞ which is contained in B_0^∞. (The existence of I_B^∞ is shown in Lemma 2 of **[6]**). If $B^\infty = H^\infty(m)$ (resp. $L^\infty(m)$), it is clear that $B_0^\infty = I_B^\infty = H_0^\infty$ (resp. $I_B^\infty = \{0\}$). In general, $I_B^\infty \subseteq H_0^\infty$ by [**6**, Lemma 2]. Let $\mathscr{L}_B^\infty$ be a self-adjoint part of B^∞, i.e. the set of all functions in B^∞ whose complex conjugates are also in B^∞.

For any subset $M \subseteq L^\infty(m)$ and $1 \leqq p < \infty$, denote by $[M]_p$ the norm closed linear span of M in $L^p(m)$ and by $[M]_*$ the weak-*closed linear span of M. For a weak-*closed superalgebra B^∞, let $B^p = [B^\infty]_p$ and let $I_B^p = [I_B^\infty]_p$ for $1 \leqq p < \infty$. For any measurable subset E of X, the function χ_E is the characteristic function of E. If $f \in L^p(m)$, denote by E_f the support set of f and by χ_f the characteristic function of E_f.