REPRESENTING CODIMENSION-ONE HOMOLOGY CLASSES BY EMBEDDED SUBMANIFOLDS

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In this paper we prove a general theorem on representing codimension-one homology classes on compact manifolds. Our theorem gives an elementary proof of the classical result that a one-dimensional homology class on a compact orientable surface can be represented by an embedded circle precisely when the class is primitive. We will call a homology class primitive if it is the zero class or if it is not a nontrivial multiple of another class.

Our representation theorem is motivated by our earlier work [2] on the classical situation. For compact orientable surfaces we developed a simple algorithm for representing primitive homology classes by embedded circles. This algorithm is also useful for proving other results related to two and three dimensional topology which are not implied by the general result here. Recently Mark Meyerson [3] has given another proof of the two dimensional case by applying Lickorish "twist" homeomorphisms to a fixed homology class.

The following theorem deals with the decomposition of a n-dimensional manifold M^n by an embedded submanifold N^{n-1} which represents a homology class $\delta \in H_{n-1}(M, Z)$. We will say that such a representation N is minimal if there is no other representation N' of δ having fewer path components.

REPRESENTATION THEOREM. *Suppose M is a compact orientable piecewise linear n-dimensional manifold and $\gamma \in H_{n-1}(M, Z)$ is a primitive nonzero class.*

1. The class $\delta = \kappa\gamma$ has a minimal representation by a submanifold.

2. If N is a minimal representation for $\delta = \kappa\gamma$ then each path component of $M - N$ has two ends and the number of path components of $N = |N|$ is κ.

Proof. The Poincaré dual to $\delta = \kappa\gamma$ can be represented by a piecewise linear mapping $P(\delta): M \to S^1$. The standard duality theorems imply that wherever $r \in S^1$ is a regular value for $P(\delta)$, then $P^{-1}(r) \subset M$ is an embedded submanifold representing δ. Hence any $\delta \in H_{n-1}(M, Z)$ has a minimal representative.

Suppose now that N is an oriented path-connected submanifold of M with $[N] \neq 0 \in H_{n-1}(M, Z)$. Since N is path connected and nontrivial