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MOMENTS OF MEASURES ON CONVEX BODIES

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In this paper the problem of moments is viewed as one of identifying a class of functions on a semigroup with a class of measures. We present integral representation theorems for linear functionals on algebras (Theorems 2.1 and 2.2) which enable us to solve moment problems for a wide class of compact sets. In particular if K is any compact convex subset of R^3 with nonvoid interior, necessary and sufficient conditions are given for a triple indexed sequence $f(n_1, n_2, n_3)$ to admit an integral representation of the form $f(n_1, n_2, n_3) = \int_K t_1^{n_1} t_2^{n_2} t_3^{n_3} d\mu(t)$ $(t = (t_1, t_2, t_3))$. Here, of course the semigroup S considered is all triples of nonnegative integers under coordinate addition. As in the case of Hausdorff's "little moment problems" the solution depends on certain linear combinations of shift operators.

We consider the three coordinate shift operators E_1, E_2, E_3 defined on the functions f on S, e.g. $(E_2 f)(n_1, n_2, n_3) = f(n_1, n_2 + 1, n_3)$. If K is the octahedron $\{t \in R^3 | 1 \pm t_1 \pm t_2 \pm t_3 \geqq 0\}$. Then f admits a necessarily unique, nonnegative representing measure if and only if $(\Delta f)(0, 0, 0) \geqq 0$, where Δ is any product of difference operators of the form $I \pm E_1 \pm E_2 \pm E_3$. The octahedron example is typical of all bounded nondegenerate convex polyhedra in that the difference operators used to describe those functions which admit representing measures are defined in terms of the facial functionals of the polyhedron K. Necessary and sufficient conditions for the existence of representing signed measures are also given in §3.

In §4 we apply Theorems 2.1 and 2.2 to algebras of shift operators on arbitrary commutative semigroups. This leads to a more general notion of functions of Bounded Variation than has previously been introduced on semigroups cf. [**6**, **11**, **12** and **14**]. The classical notion of positive definite function on a group is recast in terms of finite difference. We conclude by giving necessary and sufficient conditions for a linear functional on a commutative B^*-algebra with identity to be in the span of the positive linear functional.

2. Positive and BV-functionals on algebras with involution.

Let $\mathscr{A}$ be a real or complex commutative algebra with identity 1 and involution $*$. If $\mathscr{A}$ is real we assume $x^* = x$ for all $x \in \mathscr{A}$. Let $\mathscr{T}$ be a subset of $\mathscr{A}$ such that: