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TENSOR PRODUCTS OF FUNCTION RINGS UNDER COMPOSITION

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Let $C(X)$, $C(Y)$ be the rings of real-valued continuous functions on the completely regular Hausdorff spaces X, Y and let $T = C(X) \otimes C(Y)$ be the subring of $C(X \times Y)$ generated by functions of the form fg, where $f \in C(X)$ and $g \in C(Y)$. If P is a real polynomial, then $P \circ t \in T$ for every $t \in T$. If $G \circ t \in T$ for all $t \in T$ and if G is analytic, then G is a polynomial, provided that X and Y are both infinite (A. W. Hager, Math. Zeitschr. 92, (1966), 210–224, Prop. 3.). In this note I remove the condition of analyticity. Clearly the cardinality condition is necessary, for if either X or Y is finite, then $T = C(X \times Y)$ and $G \circ t \in T$ for every continuous G and for every $t \in T$.

It is convenient to admit a somewhat wider class of G's. Let $T^* = T + iT$, that is, the set of all functions $t_1 + it_2$ with $t_1, t_2 \in T$. (T^* is the tensor product of the complex-valued continuous function rings on X and Y). Define $K(X, Y)$ as the set of all continuous complex-valued functions G on R (the reals) with the property that $G \circ t \in T^*$ for all $t \in T$. Then the result is

THEOREM. *If X and Y are infinite completely regular Hausdorff spaces, then $K(X, Y)$ consists of all the polynomials with complex coefficients.*

It follows from the Theorem that if $G \circ t \in T$ for all $t \in T$, then G is a polynomial with real coefficients.

The proof of the Theorem, which is rather lengthy, will be broken up into a sequence of lemmas.

LEMMA 1. *Let φ and ψ be continuous mappings of X and Y onto X' and Y' respectively. Then $K(X, Y) \subset K(X', Y')$.*

Proof. Let $G \in K(X, Y)$, $t' \in T' = C(X') \otimes C(Y')$.
I must show that $G \circ t' \in T'^*$. Define t by

$$t(x, y) = t'(\varphi(x), \psi(y)) \qquad (x \in X,\ y \in Y).$$

Clearly $t \in T$, and by hypothesis $G \circ t \in T^*$. That is, there are continuous complex-valued functions $u_1, \cdots, u_n$ on X, $v_1, \cdots, v_n$ on Y, such that