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GENERALIZED PRIMITIVE ELEMENTS FOR TRANSCENDENTAL FIELD EXTENSIONS

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Let L be a finitely generated separable extension of a field K of characteristic $p \neq 0$. Artin's theorem of a primitive element states that if L is algebraic over K, then L is a simple extension of K. If L is non-algebraic over K, then an element $\theta \in L$ with the property $L = L'(\theta)$ for every L', $L \supseteq L' \supseteq K$, such that L is separable algebraic over L' is called a generalized primitive element for L over K. The main result states that if $[K : K^p] > p$, then there exists a generalized primitive element for L over K. An example is given showing that if $[K : K^p] \leqq p$, then L need not have a generalized primitive element over K.

I. Introduction. Let L be a finitely generated extension of a field K of characteristic $p \neq 0$. Artin's theorem of the primitive element states that if L is separable algebraic over K, then L is a simple extension of K. In this paper we examine the following analogue of Artin's theorem in the case where L is a separable non-algebraic extension of K. Does there exist an element $\theta \in L$ with the property that θ is a primitive element for L over every intermediate field L' such that L is separable algebraic over L'? The main result states that if K has at least two elements in a p-basis, then there does exist such a generalized primitive element (Theorem 4). Such elements θ are characterized by the condition that L is reliable over $K(\theta)$ (Theorem 1). As a corollary, it follows that automorphisms of L over K are uniquely determined by their action on a generalized primitive element θ. Other results which indicate the essential nature of a generalized primitive element include the following. If L_1 and L_2 are intermediate fields of L/K where L is separable over L_1 and L_2, then $L_2 \supseteq L_1$ if and only if some generalized primitive element for L_1 is in L_2 (Theorem 6).

II. Generalized primitive elements. Throughout we assume L is a finitely generated extension of a field K of characteristic $p \neq 0$. As usual, a relative p-basis for L over K is a minimal generating set for L over $K(L^p)$.

DEFINITION. L is a reliable extension of K if $L = K(M)$ for every relative p-basis M of L over K.