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CONTINUITY AND COMPREHENSION IN INTUITIONISTIC FORMAL SYSTEMS

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Two questions which are of fundamental importance in the foundations of constructive mathematics are

(1) Are all extensional functions (say from N^N to N) continuous?

(2) What general principles for defining sets (or species) are constructively justifiable?

This paper is concerned with metamathematical results related to these questions.

Within the framework of a formal system, we can ask if one can find any necessary relations between the answers to the two questions posed above. We show that, within the language of second-order arithmetic, one cannot find any such relations; even if one includes Church's thesis, which says that every constructive function is recursive. In earlier work, we have proved independence results related to question (1) in the context of the language of arithmetic. The main tool of the present paper is an extension of our earlier methods to second-order comprehension principles.

It is fairly easy to prove the consistency of strong principles of set existence with the continuity of extensional functions, even in the presence of Church's thesis (see discussion in **[3]**). And, as mentioned, the case where one does not have strong set existence principles has been dealt with in **[1]**. The main problem, then, is the independence of the continuity of extensional functions from strong set existence principles. Of course, if all the formal axioms considered are classically true, this independence is trivial; but we are interested in the independence in an axiomatic framework including nonclassical principles. Foremost among such principles is Church's thesis, which (in a suitable formulation) will reduce members of N^N to recursive indices, and functions from N^N to N to effective operations, which compute the function value recursively from an index of the argument. Thus, under Church's thesis CT, the statement "all functions are continuous" reduces to an arithmetical proposition about effective operations. This proposition (for the case of N^N) is called KLS, after Kreisel, Lacombe, and Shoenfield, who gave a classical proof of it **[5]**. It also happens that this sentence KLS lies in a syntactic class for which CT is conservative (over all the theories we will consider; see discussion in the text). Thus the difficult part of our problem is to prove the independence of KLS from various principles of set existence.