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CONVERGENCE THEOREMS IN BANACH ALGEBRAS

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It is shown that several convergence theorems for linear operators, usually established by Hilbert space techniques are consequences of the general relations between convergence, monotonicity and order units in ordered Banach spaces.

1. Introduction.

Hilbert space techniques have been used to investigate the convergence properties of matrix and linear operator iterative methods by Reich **[11]**, Stein **[14]**, **[15]**, Ostrowski **[7]**, John **[6]**, Householder **[4]**, **[5]**, Petryshyn **[8]**, **[9]**, **[10]** and de Pillis **[2]**, among others. It is the purpose of the present note to show that these results also follow from the general properties of nonnegative operators on ordered Banach spaces with normal cone and order units. In order to cover most of the above mentioned results without excessive technical preliminaries, we shall present our technique of proof in the framework of Banach algebras.

A direct proof of the relations between convergence, monotonicity and order units, not going through generalizations of the Perron–Frobenius theory, is proposed in §2. Needed properties of the hermitian elements of complex unital Banach algebras and of B^* algebras are established in §3. Convergence properties are considered in §4, mainly in B^* algebras.

General notations and terminology are taken from Schaefer **[13]** with the following exceptions: an element x of an ordered vector space E with cone C is called nonnegative (resp. positive) and denoted $x \geqq 0$ (resp. $x > 0$) if $x \in C$ (resp. if x is an order unit); $A \in L(E, F)$ where E and F are ordered vector spaces is called nonnegative (resp. positive) if $x \geqq 0$ implies $Ax \geqq 0$ (if $x \geqq 0$, $x \neq 0$ implies $Ax > 0$); it is called monotone if $Ax \geqq 0$ implies $x \geqq 0$.

Specific notations and terminology relative to Banach algebras are taken from Bonsall and Duncan **[1]**.

2. Monotonicity and convergence.

Convergence properties of nonnegative operators on an ordered (real) Banach space with normal cone and order units are established in the present section.