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A COMMUTATIVITY THEOREM FOR NON-ASSOCIATIVE ALGEBRAS OVER A PRINCIPAL IDEAL DOMAIN

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Let A be an algebra (not necessarily associative) over a principal ideal domain R such that for all a, $b \in A$, there exist α, $\beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$. It is shown that A is commutative.

Throughout this paper N will denote the set of natural numbers and Z^+ the set of positive integers. A will denote an algebra with identity 1 over a Principal Ideal Domain R. If a, $b \in A$ then $[a, b] = ab - ba$. If $\alpha, \beta \in R$, then (α, β) denotes the greatest common divisor of α and β. If $a \in A$, then the *order* of a, $o(a)$ is the generator of the ideal $I = \{\alpha \mid a \in R,\ \alpha a = 0\}$ of R. $o(a)$ is unique up to associates. As a generalization of concepts in **[1]**, **[2]**, **[3]**, **[4]**, **[5]** we consider the following:

(*) For all a, $b \in A$, there exist α, $\beta \in R$ such that $(\alpha, \beta) = 1$ and $\alpha ab = \beta ba$.

We will show that if A satisfies (*), then A is commutative. This generalizes [**3**; Theorem 3.5].

LEMMA 1. *Let p be a prime in R, $m \in Z^+$ such that $p^m A = (0)$. If A satisfies (*), then A is commutative.*

Proof. Let C denote the center of A. Let $x \in A$, $o(x) = p^k$, $k \in N$. We prove by induction on k that $x \in C$. If $k = 0$, then $x = 0$. So let $k > 0$. Let $y \in A$. First we show

$$[x, y] \neq 0 \quad \text{implies} \quad [yx, y] = 0. \tag{1}$$

If $yx = 0$, this is trivial. So let $yx \neq 0$. Now for some $\alpha_1, \alpha_2 \in R$,

$$\begin{aligned} &\alpha_1 xy = \alpha_2 yx, (\alpha_1, \alpha_2) = 1 \\ &\beta_1(x+1)y = \beta_2 y(x+1), (\beta_1 \beta_2) = 1. \end{aligned} \tag{2}$$

So $\alpha_1\beta_1(x+1)y = \alpha_1\beta_2 y(x+1)$. Thus substituting the above, we get

$$(\alpha_2\beta_1 - \alpha_1\beta_2)yx = (\alpha_1\beta_2 - \alpha_1\beta_1)y. \tag{3}$$