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SPACES OF SIMILARITIES IV: (s, t)-FAMILIES

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The determination of spaces of similarities is a generalization of the Hurwitz problem of compasition of quadratic forms. For forms σ, q over the field F, we write $\sigma < \mathrm{Sim}\,(q)$ if q admits composition with σ. When F is the real or complex field, the possible dimensions of σ and q were determined long ago by Radon and Hurwitz. We show that these classical bounds are still correct over any field F of characteristic not 2.

This paper deals with the more delicate question of which quadratic forms σ, q over F can admit composition. The motivation of much of this work is the Pfister factor conjecture: if q is a form of dimension 2^m, and $\sigma < \mathrm{Sim}\,(q)$ for some form σ of large dimension, then q must be a Pfister form. We prove this true in general when $m \leqq 5$, and we also prove it true for all m for a certain class of fields which includes global fields.

Introduction. This paper continues the work on similarities initiated in [11], [12]. The objects studied are nonsingular λ-forms ($\lambda = \pm 1$) over a field F of characteristic not 2. We follow the notation of [8]. The first two sections concern the following question: Given $\lambda = \pm 1$ and n, what quadratic forms σ can be realized as a subspace of $\mathrm{Sim}\,(V, B)$, for some n-dimensional λ-space (V, B)? When $n = 2^m \cdot n_0$, n_0 odd, and $\dim \sigma \geqq 2m - 1$, a complete solution is found, characterizing such forms σ in terms of the signed determinant $d_{\pm}\sigma$ and the Witt invariant $c(\sigma)$. In fact, a more general characterization of (s, t)-families on $\mathrm{Sim}\,(V, B)$ is found when $s + t \geqq 2m - 1$. In working with (s, t)-families consistently, the results are more symmetrical and easier to prove. For example, we obtain a new computation of the values of the Hurwitz functions $\rho_t^{\lambda}(n)$.

In the third section, the Pfister factor conjecture of [12; (7.1)] is restated in terms of (s, t)-families. An inductive method is then used to give new proofs of this conjecture in the cases $m = 4, 5$. This method is also used to prove the conjecture for all values of m when F is a global field.

The last two sections of this paper deal with the odd factor conjecture [12; (7.4)]. This question is settled for small families by means of a decomposition result for Pfister factors [15]. In the special case of positive definite forms over the rational numbers, the conjecture is proved for families of any size. These theorems over the rationals provide some insight into the theory of orthogonal designs [5], [6].