PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 1, 1977

A BOUNDED OPERATOR APPROACH TO TOMITA-TAKESAKI THEORY

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Let M be a von Neumann algebra. The Tomita-Takesaki theory associates with each cyclic and separating vector for M a strongly continuous one-parameter group, Δ^{it}, of unitary operators and a conjugate-linear isometric involution, J, of the underlying Hilbert space, such that $\Delta^{it}M\Delta^{-it} = M$ and $JMJ = M'$, where M' denotes the commutant of M.

The present paper has two purposes. In the first half of the paper we show that the operators Δ^{it} and J can, in fact, be associated with any fairly general *real* subspace of a complex Hilbert space, and that many of their properties, for example the characterization of Δ^{it} in terms of the K.M.S. condition, can be derived in this less complicated setting. In the second half of the paper we show, by using some of the ideas from the first half, that a simplified proof of the Tomita-Takesaki theory given recently by the second author can be reformulated entirely in terms of bounded operators, thus further simplifying it by, among other things, eliminating all considerations involving domains of unbounded operators.

1. Introduction. Our approach is motivated by the following observation. Let M be a von Neumann algebra on a Hilbert space $\mathscr{H}$, and let ω be a cyclic and separating vector for M. Let M_s denote the collection of self-adjoint elements of M and let $\mathscr{K}$ denote the closure of $M_s\omega$. Then $\mathscr{K}$ is a real subspace of $\mathscr{H}$ which can easily be shown to have the properties that $\mathscr{K} \cap i\mathscr{K} = \{0\}$ and $\mathscr{K} + i\mathscr{K}$ is dense in $\mathscr{H}$ (see Proposition 4.1). In [10] we found that the positions of $\mathscr{K}$ and $i\mathscr{K}$ were closely related to questions concerning an operator algebra and its commutant. In the present paper we find that the operators Δ^{it} and J depend only on the relative positions of $\mathscr{K}$ and $i\mathscr{K}$, and, in particular, can be defined in terms of the projections on these two subspaces. More generally, the operators Δ^{it} and J can be associated with any such real subspace, whether or not it comes from a von Neumann algebra. Because of this, it turns out that this subject is closely related to earlier work of Dixmier [4] and Halmos [7] on pairs of subspaces of Hilbert spaces. In fact, we digress at the end of the next section to show that our approach gives slightly simpler proofs of some of their main results.

In writing this paper we have addressed ourselves to those who are already familiar with the original approaches to Tomita-Takesaki