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FINITE DIRECT SUMS OF CYCLIC VALUATED p-GROUPS

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The purpose of this paper is to characterize finite direct sums of cyclic valuated p-groups in terms of numerical invariants.

Our starting point is the subgroup problem for finite abelian p-groups. It follows from the proof of Ulm's theorem that an isomorphism f between two subgroups of a finite abelian p-group A is induced by an automorphism of A exactly when f preserves heights in A. This suggests a device for dealing with such subgroups—we give each element of a subgroup a value, namely, its height in the containing group. Now the containing group is dispensed with and we work only with the subgroup and its valuation, which we call a valuated p-group. Theorem 32 of [2] shows that all of the information pertaining to the original embedding is essentially captured by this process. There is a natural definition of a morphism of valuated p-groups, yielding the category $\mathscr{F}_p$ of finite valuated p-groups. We then observe that $\mathscr{F}_p$ has direct sums and set about characterizing the simplest objects in $\mathscr{F}_p$, the cyclics and their direct sums. The invariants for finite abelian p-groups are provided by functors from the category of abelian groups to the category of vector spaces, each functor picking out the number of cyclic summands of a given order in any decomposition of the group into cyclics. We carry out a parallel program for direct sums of cyclics in $\mathscr{F}_p$, obtaining in Theorem 2 a complete set of invariants. An example is then given to show that the objects of $\mathscr{F}_p$ are not all direct sums of cyclics. Thus arises the question of finding some criterion for a valuated p-group to be a direct sum of cyclics. We provide such a criterion in Theorem 3, making use of the functional invariants used for direct sums of cyclics. The paper concludes with a proof that p^2-bounded valuated groups in $\mathscr{F}_p$ are direct sums of cyclics. This bound is the best possible, as the example referred to above is bounded by p^3.

Valuations. Once and for all, fix a prime p. All groups considered will be finite abelian p-groups. Let A be such a group. Define $p^0A = A$ and $pA = \{pa: a \in A\}$. Clearly pA is a subgroup of A. The subgroups p^nA are defined inductively in the obvious way. The *height* $h(a)$ of a nonzero element a of A is defined by $h(a) = n$ where $a \in p^nA \backslash p^{n+1}A$. Set $h(0) = \infty$. We agree that $\infty < \infty$ and