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ON FINITE REGULAR RINGS

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Several new properties are derived for von Neumann finite rings. A comparison is made of the properties of von Neumann finite *regular* rings and unit regular rings, and necessary and sufficient conditions are given for a matrix ring over a regular ring to be respectively von Neumann finite or unit regular. The converse of a theorem of Henriksen is proven, namely that if $R_{n\times n}$, the $n \times n$ matrix ring over ring R, is unit regular, then so is the ring R. It is shown that if $R_{2\times 2}$ is finite regular then $a \in R$ is unit regular if and only if there is $x \in R$ such that $R = aR + x(a^{\circ})$, where a° denotes the right annihilator of a in R.

1. Introduction. In [13], Henriksen posed the question whether a finite regular ring is unit regular. This was subsequently proven in part by Ehrlich [4] for a particular class of regular rings. An example of finite regular rings which are not unit regular was recently given by Bergman (1974) (see Handelman [8]). In his paper [8], Handelman showed that a regular ring R is unit regular if and only if, for any finitely generated projective right R-modules A, B, and C, $A \oplus B \cong A \oplus C$ implies $B \cong C$. He also characterized unit regular rings by perspectivity on the lattices of their principal right ideals. The purpose of this paper, however, is to characterize finite regular rings and to compare their properties with unit regular rings. Some of the results of the theory of generalized inverses [1] are used to show that in a regular ring, the properties of finiteness and unit-regularity each correspond to a suitable cancellation law for principal ideals. These cancellation laws are closely related to the substitution property of Fuchs [5], and the cancellation law of Ehrlich [4]. We shall use a result by Vidav [15] to show that if the matrix ring $R_{n\times n}$ is unit regular then so is ring R. Let us begin by defining our concepts and by stating some useful general results. A ring R is called *regular* if for all $a \in R$, there is a solution $a^- \in R$ to the equation $axa = a$. The element a^- is called as inner or 1-inverse of a [1]. Similarly, any solution to $axa = a$, $xax = x$ is called a reflexive or $1-2$ inverse of a, and will be denoted by a^+. For example, a^-aa^- is always such a solution. An element $a \in R$ is said to have a group inverse $a^{\#} \in R$ if it is a group member, i.e., a belongs to some multiplicative group of R. Necessary and sufficient conditions for $a^{\#}$ to exist are that $a^2R = aR$, $Ra^2 = Ra$, or $axa = a$, $xax = x$, $ax = xa$, for some $x \in R$ [10]. A regular ring R with unity is called *unit regular*