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ON SUBDIRECTLY IRREDUCIBLE COMMUTATIVE SEMIGROUPS

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We give a characterization of finitely generated commutative semigroups which are finitely subdirectly irreducible.

We call a semigroup S finitely subdirectly irreducible, or just irreducible, if it has the following property: when S is a subdirect product of finitely many semigroups S_i, one of the projections $S \to S_i$ must be an isomorphism. These semigroups are of interest because every finitely generated commutative semigroup is a subdirect product of finitely many irreducible semigroups. (The factors in this decomposition are then finitely generated and commutative as well as irreducible.)

Let S be a finitely generated commutative semigroup. If S is irreducible then S is either cancellative or a nilsemigroup or what we call a subelementary semigroup, i.e., the disjoint union $S = N \cup C$ of a nilsemigroup N which is also an ideal of S, and a subsemigroup $C \neq \varnothing$ every element of which is cancellative in S. The first two cases are easily dealt with and our new results are in the subelementary case. This case reduces to the other two if N or C is trivial. If $S = N \cup C$ is subelementary with N, C nontrivial, then the irreducibility of S is equivalent to four simple conditions of an elementary nature. A second characterization is also given, as follows. If S is subelementary, then S can be completed to an elementary semigroup (=a subelementary semigroup whose cancellative part C is a group); this does not affect irreducibility. Elementary semigroups can in turn be constructed by coextension techniques; in the case under consideration the construction is in terms of a group, a finite nilsemigroup and a factor set. A characterization of irreducible semigroups is also given in terms of this construction.

These results specialize immediately to a characterization of finitely generated commutative semigroups which are subdirectly irreducible. These semigroups must be finite and classify into groups, nilsemigroups and elementary semigroups. In the elementary case, our two characterizations and construction from groups and nilsemigroups are still new. In particular they go far deeper than Schein's results in [14], as one could expect since our results are also considerably less general.