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CERTAIN HYPOTHESES CONCERNING L-FUNCTIONS

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Some conditional results are discussed concerning Dirichlet L-functions. In particular, a method is introduced which on the one hand gives a new proof of a result of Wolke concerning the least prime quadratic residue, and on the other hand, gives a result on the least quadratic non-residue which does not seem to follow from previously known arguments.

Let q be an odd prime, $\chi(n)$ the Legendre symbol (n/q) and $L(s, \chi)$ the Dirichlet L-function pertaining to χ. Let Y_+ (respectively Y_-) denote the least prime p such that $\chi(p) = 1$ (respectively -1). Various results are known connecting together:

(A) zero-free regions for $L(s, \chi)$

(B) the magnitude of $L(1, \chi)$

(C) the magnitudes of Y_+ and Y_-.

Roughly speaking, a statement about any of these implies a corresponding statement about the subsequent ones.

In the case of (A) $\Rightarrow$ (B), we have the following theorem of Littlewood [8].

(1) Assume $L(s, \chi) \neq 0$ for $\sigma = \mathrm{Re}\, s > 1 - \theta(q)$. There exist positive absolute constants c_1 and c_2 such that

$$\frac{c_1\theta(q)}{\log\log q} < L(1, \chi) < \frac{c_2 \log\log q}{\theta(q)} .$$

Actually, Littlewood proves this only for $\theta(q) = 1/2$ (the Extended Riemann Hypothesis) but his method extends easily (as remarked by Elliott [4] for the lower bound) to give the stated result. A brief sketch of the method is given in Lemma 11.

In the case (A) $\Rightarrow$ (C), there is the result of Rodosskii [9]:

(2) Let $\psi > e$ and assume $L(s, \chi) \neq 0$ for $\sigma > 1 - \psi/\log q$. There is a positive constant c such that

$$Y_- \ll q^{c \log \psi/\psi} .$$

(Actually, Rodosskii's assumption is somewhat weaker, postulating a zero-free region only up to a certain height.)

In the case of the Extended Riemann Hypothesis ($\psi = 1/2 \log q$), we have the slightly stronger result of Ankeny [1]