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A SOBOLEV SPACE AND A DARBOUX PROBLEM

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This paper deals with a class of functions which are defined in an n-dimensional rectangle and which possess there, only the generalized partial derivatives of mixed type. It is shown that (i) this class contains as a proper subset the usual Sobolev class of order n, the dimension of the domain and (ii) this class can be imbedded in the space of continuous functions. In addition to the compactness of the imbedding operator, the closedness of certain nonlinear partial integro differential operators is also studied. Finally, a system of partial integro differential equations with Darboux type boundary data in a rectangle, is shown to have solutions in this class. The results of this paper are used in certain existence theorems of optimal control theory.

1. Introduction. In recent studies on existence theorems for optimization problems involving Darboux type side conditions, it was found useful and necessary to introduce a special class of functions, (see for example [11] and [7]). This special class which we shall denote as $W_p^*(G)$, $G \subset E^n$, consists of all functions $z(t)$, $t \in G$, with $z \in L_p(G)$, and such that z has all (and only) the mixed partial generalized derivatives $D_\alpha z$ of orders upto and including n (the number of independent variables) with $D_\alpha z \in L_p(G)$; thus, derivatives of z taken more than once with respect to any of the variables $t_1, \cdots, t_n$, may not even exist. In the case of $n = 2$, for example, with independent variables x and y, this would mean that for $z \in W_p^*(G)$, the generalized partials z_x, z_y and z_{xy} exist and belong to $L_p(G)$ while the pure partials z_{xx} and z_{yy} need not even exist. This class is thus analogous to the classical $C^*(G)$ where only the derivatives z_x, z_y and z_{xy} exist and are continuous.

Clearly, $W_p^*(G)$ contains $W_p^n(G)$, the usual Sobolev class of functions for which all the generalized partial derivatives of order upto and including n exist and belong to $L_p(G)$. We shall show in no. 2 below that there exist functions in $W_p^*(G)$ which are not in $W_p^n(G)$. One of the purposes of this paper is to analyze this special class of functions $W_p^*(G)$ for $G = [a, a + h] \subset E^n$, $n \geqq 1$ and in particular to show that it can be imbedded in $C(G)$, the space of continuous functions, for all p, $1 \leqq p \leqq \infty$. In particular, it follows that $W_p^n(G) \subset C(G)$ even for $p = 1$. We shall also study criteria for the compactness of the imbedding operator. In the same context, the closedness of some nonlinear integro differential operators is investigated, a result used in the closure theorems relevant to the