PACIFIC JOURNAL OF MATHEMATICS
Vol. 69, No. 2, 1977

COBORDISM CLASSES OF FIBER BUNDLES

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This paper treats the problem of determining which unoriented cobordism classes have a representative which is the total space of a fiber bundle over a sphere. We are looking for a necessary and sufficient condition for a closed, compact, differentiable manifold to be cobordant to the total space of a fiber bundle over S^k.

Our results on bundles over S^4 and S^8 extend the results of P. E. Conner [3, 4], E. E. Floyd [4], R. O. Burdick [2], W. D. Neumann [6], R. L. W. Brown [1] and R. E. Stong [7].

The following definition will facilitate the discussion.

DEFINITION 0.1. If α represents an unoriented cobordism class, we say that α fibers over S^k if α contains a representative which is the total space of a fiber bundle over S^k.

It has been shown (see [4] and [1]) that $[M^n]$ fibers over S^1 if and only if $\langle w_n, [M^n]\rangle$, which we will abbreviate by $w_n[M^n]$, is zero, and $[M^n]$ fibers over S^2 if and only if

$$\begin{cases} w_n[M^n] = 0 & \text{if } n \text{ is even} \\ w_{n-2}w_2[M^n] = 0 & \text{if } n \text{ is odd}\,. \end{cases}$$

The last condition is also sufficient for the cobordism class of a manifold to fiber over any particular N^2, since Stong has shown that if $[M^n]$ fibers over S^k, then it fibers over any manifold of dimension less than or equal to k (see [7]).

Our main results are as follows.

THEOREM I. *There are generators of $\mathfrak{N}_*$ which fiber over S^4 in all dimensions greater than or equal to* 8 *except* 11 *and, of course, those of the form* $2^j - 1$.

COROLLARY I. *We can choose generators of $\mathfrak{N}_* = Z_2[x_i \mid i \neq 2^j - 1]$ so that an element of either one of the following subalgebras will fiber over S^4 if and only if the Stiefel-Whitney numbers associated with w_n and $w_{n-2}w_2$ are both zero. The subalgebras are*:

$$I = Z_2[x_i \mid i \neq 5, 6, 11 \quad \text{or} \quad 2^j - 1]$$
$$J = Z_2[x_i \mid i \neq 4, 5, 11 \quad \text{or} \quad 2^j - 1]\,.$$

THEOREM II. *No indecomposable* 11-*dimensional class fibers*