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MULTISECTIONED PARTITIONS OF INTEGERS

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This paper presents identities on generating functions for multisectioned partitions of integers by developing in the language of partitions some powerful and essentially combinatorial techniques from the literature of principal differential ideals. D. Mead has stated in Vol. 42 of this journal that one can obtain interesting combinatorial relations by constructing different vector space bases for a subspace of a differential ring and using the fact that the cardinality of all bases is the same. The results of the present paper are of this nature.

In particular, we enumerate certain sets of ordered pairs of generalized tableaux that have a central role in Mead's paper. Tableaux were used by A. Young and others to study the structure of the symmetric groups S_n. In [3], D. Knuth used an "insertion into tableau" construction of C. Schensted to give a direct 1-to-1 correspondence between "generalized permutations" and ordered pairs of "generalized Young tableaux" having the same shape. In [5], Mead independently proved the existence of such a bijection while developing a new vector space basis for the ring of differential polynomials in n independent differential indeterminates. Mead's paper deals with principal differential ideals generated by Wronskians and used determinantal identities going back to Cayley. The ordered pairs of generalized tableaux used by Mead appear in a more general setting in the paper [1] by Doubilet, Rota, and Stein.

1. Multisectioned partitions. Let w be a nonnegative integer. Here a *partition of w into d parts*, or of *degree d*, is a d-tuple $(p_1, \cdots p_d)$ of nonnegative integers p_k with $p_1 \leqq p_2 \leqq \cdots \leqq p_d$ and $w = p_1 + \cdots + p_d$. If $d = 0$, we agree that there is one (ideal) partition of 0 but no partition of any $w > 0$.

Let $P = (P_1, \cdots, P_n)$, where P_i is a partition of w_i of degree d_i, and let $w = w_1 + \cdots + w_n$; then P is an *n-section partition of w with signature* $D = [d_1, \cdots, d_n] = \operatorname{sig} P$ and the *weight* of P is w.

For applications to differential algebra, it is convenient to use two rowed matrices

$$(1) \qquad M = \begin{pmatrix} i_1 & i_2 \cdots i_d \\ j_1 & j_2 \cdots j_d \end{pmatrix}$$

in which $i_k \in \{1, 2, \cdots, n\}$, $j_k \in \{0, 1, \cdots\}$, $i_k \leqq i_{k+1}$ for $1 \leqq k < d$, and