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A GENERALIZATION OF CARISTI'S THEOREM WITH APPLICATIONS TO NONLINEAR MAPPING THEORY

DAVID DOWNING AND W. A. KIRK

Suppose X and Y are complete metric spaces, $g: X \to X$ an arbitrary mapping, and $f: X \to Y$ a closed mapping (thus, for $\{x_n\} \subset X$ the conditions $x_n \to x$ and $f(x_n) \to y$ imply $f(x) = y$). It is shown that if there exists a lower semicontinuous function φ mapping $f(X)$ into the nonnegative real numbers and a constant $c > 0$ such that for all x in X, $\max\{d(x, g(x)), cd(f(x), f(g(x)))\} \leqq \varphi(f(x)) - \varphi(f(g(x)))$, then g has a fixed point in X. This theorem is then used to prove surjectivity theorems for nonlinear closed mappings $f: X \to Y$, where X and Y are Banach spaces.

1. **Introduction.** The following fact is well-known in the theory of linear operators;

(1.1) *Let X and Y be Banach spaces with D a dense subspace of X, and let $T: D \to Y$ be a closed linear mapping with dual T'. Suppose the following two conditions hold:*

(i) $N(T') = \{0\}$.

(ii) *For fixed* $c > 0$, $\operatorname{dist}(x, N(T)) \leqq c \|Tx\|$, $x \in D$.

Then $T(D) = Y$.

Proof. Because T is a closed mapping it routinely follows from (ii) that $T(D)$ is closed in Y (e.g., [15, p. 72]), whence it follows from the Hahn-Banach theorem (cf. [17, p. 205]) that $(N(T'))^{\perp} = T(D)$ where $(N(T'))^{\perp}$ denotes the annihilator in Y of the nullspace of T'. By (i), $(N(T'))^{\perp} = Y$.

It is our objective in this paper to give a nonlinear generalization of the above along with more technical related results. The key to our approach is an application of a new generalized version of Caristi's fixed point theorem. While our method parallels that of Kirk and Caristi [12], these new results differ from those of [12] and the earlier 'normal solvability' results of others, e.g., Altman [1], Browder [3-6], Pohozhayev [13, 14], and Zabreiko-Krasnoselskii [18], in that by using the improved fixed point theorem we are able to replace the usual closed range assumption with the assumption that the mapping be closed (in conjunction with a condition which in the linear case reduces to (ii)). Before doing this, however, we state and prove our fixed point theorem.