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BICONTRACTIVE PROJECTIONS AND REORDERING OF L_p-SPACES

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On a Banach space we call a projection, P, bicontractive, if $\|P\| \leqq 1$ and $\|I-P\| \leqq 1$. In this paper we completely describe bicontractive projections on an L_p-space ($1 \leqq p < \infty$) by showing that for every such bicontractive projection P, $2P-I$ is an involutive linear isometry. Duality then gives the same result for pre-dual L_1-spaces (in particular for M-spaces). The analysis of bicontractive projections is used, with $p \neq 2$, to describe all Banach lattices which are linearly isometric to an L_p-space.

Such projections on $L_p(\mu)$, when $1 < p < \infty$, $p \neq 2$, and μ is a probability measure, have been considered by Byrne and Sullivan [2]. Their analysis gave the basic result, that $2P - I$ is an isometry. Their methods are different from ours and depend heavily on Lamperti's description [6] of isometries of L_p-spaces; and their approach is weighted much more towards independence of sub σ-algebras rather than the isometry property. Some minor changes in the formulation of their results were made later in Byrne's 1972 Ph. D dissertation at the University of Pittsburg. Our approach relies on our earlier complete description [1] of contractive projections on an L_p-space. We include, in § 3, a rapid survey of some of the Byrne, Sullivan results where their approach is different and outline very simple deductions of their results from ours.

The question of Banach lattice orderings of L_p, under the usual norm, have been considered, with $1 \leqq p < \infty$ and $p \neq 2$, for the separable case by Lacey and Wojtaszczyk [5]. Their results also depend on the Lamperti isometry results and crucially on separability. Our analysis uses our previous discussion of contractive and bicontractive projections and gives a complete generalization of their work.

Throughout the paper we assume $1 \leqq p < \infty$ and $p \neq 2$. We will write $L_p = L_p(X, \Sigma, \mu)$ for the standard real or complex L_p space determined by a set X, a σ-ring, Σ, of subsets of X and a measure μ on Σ. If $f \in L_p$, $S(f) = \{t \in X: f(t) \neq 0\}$, as in [1] the ambiguity of a set of measure zero is irrelevant. Where our results are true for either choice of scalar field the field will not be specified. Where the scalar field is specified the result will be true only for the specified choice. The case $p = 2$ is omitted because the theorems we prove are all trivially true, or trivially false, in this case.