PACIFIC JOURNAL OF MATHEMATICS
Vol 70, No. 1, 1977

ATTAINING THE SPREAD AT CARDINALS OF COFINALITY ω

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Let λ be a singular cardinal of cofinality ω. We investigate the question: does every Hausdorff space with spread λ have a discrete subspace of cardinality λ? The answer is "yes" if $\lambda > 2^{\aleph_0}$ or if $\lambda < 2^{\aleph_0}$ and MA holds; however, for $\lambda < 2^{\aleph_0}$ an answer of "no" is consistent with the axioms of set theory. The proof involves showing the equivalence of the question with one about category in the real line. Similar results hold for the width of a space.

0. Introduction. All spaces are assumed Hausdorff, 2^ω is understood to have the usual product topology, and an ordinal is the set of its predecessors. The ordinal 0 is sometimes slashed ($\emptyset$). c is the cardinal $2^{\aleph_0}$.

The spread of a space X ($s(X)$) is the supremum of the cardinalities of its discrete subspaces. When $s(X)$ is a limit cardinal, we may ask whether the spread is attained — i.e., whether X has a discrete subspace of size $s(X)$. If λ is a singular strong limit cardinal spread is attained (Hajnal–Juhász; see **[1]**, Theorem 3.2); however, if $\omega < cf(\lambda) \leqq \lambda \leqq c$ spread is not (**[3]**). For results when λ is a regular limit cardinal, see **[1]**, p. 40. When $cf(\lambda) = \omega$, spread is always attained provided X is strongly T_2 (a condition between T_2 and T_3) (Hajnal–Juhász; see **[1]**, Theorem 3.3); however, the general problem for all T_2 spaces has remained open. We use $SA(\lambda)$ to abbreviate the assertion: the spread is attained in all T_2 spaces of spread λ.

Recall that a set $Y \subseteq 2^\omega$ is nowhere dense (n.w.d.) iff the closure of Y has empty interior, and Y is first category iff Y is a countable union of n.w.d. sets. We let $L(\lambda)$ be the assertion: for all $X \subseteq 2^\omega$ of cardinality λ, there is a $Y \subseteq X$ of cardinality λ such that Y is first category in 2^ω. Thus, if $\lambda > c$, $L(\lambda)$ holds vacuously. Under MA, $\neg L(c)$ (Luzin **[2]**), but $L(\lambda)$ holds for all $\lambda < c$. It is consistent that $L(\lambda)$ holds for all λ (add ω_2 random reals to a model of CH). It is also consistent with c being arbitrarily large that $L(\lambda)$ fails for all uncountable $\lambda \leqq c$. To see this, add a sequence of Cohen reals, $\langle r_\xi : \xi < \kappa \rangle$; then for $\lambda \leqq \kappa$, $\{r_\xi : \xi < \lambda\}$ is in fact a Luzin set—i.e., all first category subsets are countable. It is clear from these remarks that all the results stated in the abstract follow from:

MAIN THEOREM. *If $\omega = cf(\lambda) < \lambda$, then $SA(\lambda)$ iff $L(\lambda)$.*