PACIFIC JOURNAL OF MATHEMATICS
Vol. 70, No. 1, 1977

ON THE MEASURABILITY OF CONDITIONAL EXPECTATIONS

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It is shown that for a measurable stochastic process V and a nondecreasing family of σ-algebras $\mathcal{A}_t$ there exists a measurable stochastic process V^* such that $V^*(t,\cdot)$ is a version of $E(V(t,\cdot)|\mathcal{A}_t)$ for all t.

Let $(\Omega, \mathcal{A}, P)$ be a probability space (not necessarily complete), T an interval (bounded or unbounded) of the real line and V a real-valued stochastic process defined on $T \times \Omega$ which is a measurable process, see Doob [3, p. 60]. Let $\mathcal{A}_t$, $t \in T$, $\mathcal{A}_t \subset \mathcal{A}$ form a nondecreasing family of σ-algebras. We shall prove in this note that under some boundedness condition on V the conditional expectations with respect to P, $E(V(t,\cdot)|\mathcal{A}_t)$ can be chosen as to define a measurable process on $T \times \Omega$. A similar statement appears in a paper by Brooks [1] but there it is additionally assumed that the family of σ-algebras is left-continuous, and the proof given there does not seem to carry over to a general nondecreasing family.

THEOREM. *Suppose for each* $t \in T$: $V(t,\cdot) \geqq 0$ *P-a.s. or* $\int |V(t,\cdot)|\,dP < \infty$. *Then there exists a measurable process* V^* *such that for each* $t \in T$, $V^*(t,\cdot)$ *is a version of* $E(V(t,\cdot)|\mathcal{A}_t)$.

Proof. Since for any $t \in T$

$$E(V(t,\cdot)|\mathcal{A}_t) = E(V(t,\cdot)^+|\mathcal{A}_t) - E(V(t,\cdot)^-|\mathcal{A}_t)$$

we may assume without loss of generality that for each $t \in T$ $V(t,\cdot) \geqq 0$ *P-a.s.* Using the linearity and monotone convergence property of conditional expectations the theorem now is easily reduced to the case that V is the characteristic function I_D of some subset $D = B \times A$ of $T \times \Omega$ with $A \in \mathcal{A}$ and B belonging to the Borel sets of T.

Since $E(I_D(t,\cdot)|\mathcal{A}_t) = I_B(t)E(I_A|\mathcal{A}_t)$ holds it is enough to show that $E(I_A|\mathcal{A}_t)$ can be chosen to form a measurable process. Let $\mathcal{M}$ denote the set of all random variables on $(\Omega, \mathcal{A}, P)$ taking values in $[0,1]$ with random variables that are equal *P-a.e.* identified. Then $\mathcal{M}$ is a metrizable topological space under the topology of convergence in