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LOCALLY BOUNDED TOPOLOGIES ON $F(X)$

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It is classic that, to within equivalence, the only valuations on the field $F(X)$ of rational functions over a field F that are improper on F are the valuations v_p, where p is a prime of the principal ideal subdomain $F[X]$ of $F(X)$, and the valuation v_∞, defined by the prime X^{-1} of the principal ideal subdomain $F[X^{-1}]$ of $F(X)$ ([1], p. 94, Corollary 2). If $\mathcal{T}$ is the supremum of finitely many of the associated valuation topologies, then $\mathcal{T}$ is a Hausdorff, locally bounded ring topology on $F(X)$ for which F is a bounded set and for which there is a nonzero topological nilpotent a. In this paper we shall show conversely that any topology on $F(X)$ having these properties is the supremum of finitely many valuation topologies.

A subset S of a topological ring R is *bounded* if given any neighborhood V of 0, there exists a neighborhood U of 0 such that $SU \subseteq V$ and $US \subseteq V$. The topology on R is *locally bounded* if there is a bounded neighborhood of 0. A bounded subfield of a Hausdorff topological ring is discrete ([2], p. 119, Exercise 13). It is easy to see that if $\lim_{n\to\infty} x_n = 0$ and if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence, then $\lim_{n\to\infty} a_n x_n = 0$.

An element c of a topological ring is a *topological nilpotent* if $\lim_{n\to\infty} c^n = 0$. Let $\mathcal{P} = \{p \in F[X]: p$ is a prime polynomial$\}$, and let $\mathcal{P}' = \mathcal{P} \cup \{\infty\}$. For each $p \in \mathcal{P}'$, we shall denote by $\mathcal{T}_p$ the topology defined by the valuation v_p. Then for any finite subset L of $\mathcal{P}'$, $\sup_{p\in L} \mathcal{T}_p$ has a nonzero topological nilpotent. Indeed, let g be the product of the members of $L \cap \mathcal{P}$. If $\infty \notin L$, g is a nonzero topological nilpotent for $\sup_{p\in L} \mathcal{T}_p$; if $\infty \in L$, let q be a prime polynomial not in L and let $r > 0$ be such that $\deg(q^r) > \deg g$; then $g q^{-r}$ is a nonzero topological nilpotent of $\sup_{p\in L} \mathcal{T}_p$.

We recall that a *norm* $\|\cdot\|$ on a field K is a function to the nonnegative reals satisfying $\|x\| = 0$ if and only if $x = 0$, $\|x - y\| \leqq \|x\| + \|y\|$, and $\|xy\| \leqq \|x\|\,\|y\|$ for all $x, y \in K$. Clearly a subset of K is bounded in norm if and only if it is bounded for the topology defined by the norm; in particular the topology given by a norm is a locally bounded topology. We shall use the following theorem of P. M. Cohn ([4], Theorem 6.1): A Hausdorff, locally bounded ring topology on a field K for which there is a nonzero topological nilpotent is defined by a norm.

THEOREM 1. *Let F be a field and x a transcendental element over F in some field extension. Let $\mathcal{T}$ be a Hausdorff, locally bounded ring*