PACIFIC JOURNAL OF MATHEMATICS
Vol 70, No 2, 1977

UNBOUNDED REPRESENTATIONS OF $*$-ALGEBRAS

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Basic results on unbounded operator algebras are given, a general class of representations, called adjointable representations is introduced and irreducibility of representations is considered. A characterization of self-adjointness for closed, strongly cyclic $*$-representations is presented.

1. Introduction. Algebras of unbounded operators and unbounded representations of $*$-algebras have been important in quantum field theory [**1**, **3**, **9**, **10**] and certain studies of Lie algebras [**5**, **7**]. The present paper proceeds along the lines initiated and developed by Robert Powers [**6**, **7**] and much of the notation and definitions follow [**6**]. In §2, we present some basic results concerning unbounded operator algebras, introduce a class of representations called adjointable representations, and consider irreducibility of representations. Section 3 characterizes the self-adjointness of closed, strongly cyclic $*$-representations.

2. Adjointable representations. Let M and N be subspaces (linear manifolds) in a Hilbert space H. Let $L(M,N)$ and $L_c(M,N)$ denote the collection of linear operators and closable linear operators, respectively with domain M and range in N. For simplicity we use the notation $L(M)=L(M,M)$ and $L_c(M)=L_c(M,M)$. Notice that $L_c(H)$ is the set of bounded linear operators on H. We denote the domain of an operator A by $D(A)$ and if A is closable we denote the closure of A by $\bar{A}$. A collection of operators $\mathscr{B}$ is an *op-algebra* if there exists a subspace M such that $\mathscr{B}\subseteq L(M)$ and $A,B\in\mathscr{B}$ implies $AB,(\alpha A+B)\in\mathscr{B}$ for all $\alpha\in\mathbf{C}$. A set $\mathscr{B}\subseteq L(M)$ is *symmetric* if M is dense and $A\in\mathscr{B}$ implies $D(A^*)\supseteq M$ and $A^*\,|\,M\in\mathscr{B}$. A symmetric op-algebra $\mathscr{B}\subseteq L(M)$ that contains $I\,|\,M$ is called an *op*-algebra.* It is easy to see that if $\mathscr{B}\subseteq L(M)$ is an op*-algebra, then the map $A\to A^*\,|\,M$ is an involution so $\mathscr{B}$ is a $*$-algebra. Also, if π is a representation of a $*$-algebra $\mathscr{A}$, then $\pi(\mathscr{A})=\{\pi(A): A\in\mathscr{A}\}$ is an op-algebra and if π is a $*$-representation of $\mathscr{A}$, then $\pi(\mathscr{A})$ is an op*-algebra (we always assume that a $*$-algebra contains an identity I).

A set $\mathscr{B}\subseteq L(M,N)$ is *directed* if for any $B_1,B_2\in\mathscr{B}$ there exists a $B_3\in\mathscr{B}$ such that $\|B_1x\|,\|B_2x\|\leqq\|B_3x\|$ for all $x\in M$. For example, if $\mathscr{B}\subseteq L_c(H)$ and $\{\lambda I:\lambda\geqq 0\}\subseteq\mathscr{B}$, then $\mathscr{B}$ is directed. Indeed, just let $B_3=(\|B_1\|+\|B_2\|)I$. For an example of an unbounded directed set, let $\mathscr{B}\subset L(M,H)$ and suppose $B_1,B_2\in\mathscr{B}$ implies $B_3=$