PACIFIC JOURNAL OF MATHEMATICS
Vol. 70, No. 2, 1977

ON SOME NEW GENERALIZATIONS OF SHANNON'S INEQUALITY

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Let $A_n = \{P \in R^n : P = (p_1, p_2, \cdots, p_n)$, where $\sum_{i=1}^n p_i = 1$ and $p_i > 0$ for $i = 1, 2, \cdots, n\}$ and let $B_n = \{P \in A_n : p_1 \geqq p_2 \geqq \cdots \geqq p_n\}$. We show that the inequality

$$(1) \qquad \sum_{i=1}^n p_i f(p_i) \geqq \sum_{i=1}^n p_i f(q_i)$$

for all $P, Q \in B_n$ and some integer $n \geqq 3$, implies that $f(p) = c \log p + d$, where c is an arbitrary nonnegative number and d is an arbitrary real number. We show, furthermore, that if we restrict the domain of the inequality (1) to those $P, Q \in B_n$ for which $P > Q$ (Hardy–Littlewood–Pólya order), then any function that is convex and increasing satisfies (1).

1. Let $P, Q \in A_n$. Then the inequality

$$(2) \qquad \sum_{i=1}^n p_i \log p_i \geqq \sum_{i=1}^n p_i \log q_i$$

holds with equality iff $P = Q$ **[9]**. The inequality (2) has numerous applications in information theory **[1]**. Conversely, it was proved in **[3]**, that the so-called Shannon-inequality

$$(3) \qquad \sum_{i=1}^n p_i f(p_i) \geqq \sum_{i=1}^n p_i f(q_i)$$

for all $P, Q \in A_n$ and some integer $n \geqq 3$, implies that

$$f(p) = c \log p + d \quad \text{for} \quad p \in (0, 1)$$

where c is some nonnegative number and d is some real number.

The inequality (3) has other interpretations, too. Let us mention the following. Let $E_1, E_2, \cdots, E_n$ be a mutually exclusive and complete system of the events of an experiment with the probability distribution $(p_1, p_2, \cdots, p_n)$ with positive probabilities. Let $q_1, q_2, \cdots, q_n$ be the estimates of these probabilities ($q_i > 0$ for $i = 1, \cdots, n$). If the ith event