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CURVATURE FUNCTIONS ON LORENTZ 2-MANIFOLDS

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0. Introduction. This paper deals with the problem of describing those functions which can arise as Gaussian curvatures on 2-dimensional Lorentz manifolds, specifically, the 2-dimensional torus T^2 and the plane R^2. It is well known that the only compact connected oriented 2-dimensional manifold which admits a Lorentz metric is the torus T^2, so restricting attention to T^2 represents no loss in generality.

The technique used is that used by Kazdan and Warner in their study of Gaussian curvatures on Riemannian manifolds. The manifold M is given the standard flat Lorentz metric $g : ds^2 = dx^2 - dy^2$. Then a given smooth function K will be the Gauss curvature of the Lorentz metric $\bar{g} = e^{2u}g$ if the function u is a smooth solution of the nonlinear hyperbolic partial differential equation $\Box u = u_{xx} - u_{yy} = -Ke^{2u}$. Solving this equation on T^2 is equivalent to finding a solution $u(x, y)$ in the plane which is periodic in each variable. Although a considerable literature exists on the problem of global periodic solutions of nonlinear hyperbolic partial differential equations, the emphasis is on other types of equations, and our results appear to be new. Our results illustrate a number of significant differences between curvature functions of Riemannian (positive definite) metrics and Lorentz metrics.

Following Kazdan and Warner [**7, 8**], we say that metrics $\bar{g}$ and g on a manifold M are *pointwise conformal* if $\bar{g} = e^{2u}g$ for some smooth function u on M and that $\bar{g}$ and g are *conformally equivalent* if there is a diffeomorphism ϕ of M and a smooth function u such that $e^{2u}g$ is the metric obtained by pulling back $\bar{g}$ under ϕ, i.e. $\phi^*(\bar{g}) = e^{2u}g$. We prescribe a Lorentz metric g on the manifold M and attempt to realize a given function K as the curvature of a Lorentz metric $\bar{g}$ which is pointwise conformal to g or, if that is not possible, which is conformally equivalent to g. This approach leads to the problem of solving the nonlinear hyperbolic partial differential equation $\Delta u = -k + Ke^{2u}$, where k and Δ are the Gaussian curvature and Laplace–Beltrami operator, respectively, in the given metric g. (For a derivation of this equation in local coordinates, see Eisenhart [**3**, p. 90].) The problem of showing that K is the curvature of a metric $\bar{g}$ conformally equivalent to g is precisely that of finding a diffeomorphism ϕ of M such that one can solve $\Delta u = -k + (K \circ \phi)e^{2u}$. For the flat Lorentz metric $g: ds^2 = dx^2 - dy^2$, $\Delta u = -\Box u = -(u_{xx} - u_{yy})$ and $k = 0$, so the equation in question is $\Box u = u_{xx} - u_{yy} = -Ke^{2u}$.