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PERMUTATIONS OF THE POSITIVE INTEGERS WITH RESTRICTIONS ON THE SEQUENCE OF DIFFERENCES

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Let $\{a_k\}$ be a sequence of positive integers and $d_k = |a_{k+1} - a_k|$. We say that $\{a_k\}$ is a permutation if every positive integer appears once and only once in the sequence, $\{a_k\}$. We prove the following: Let $\{m_i\}$ be any sequence of positive integers, then there exists a permutation $\{a_k\}$ such that $|\{k|d_k = i\}| = m_i$.

By a permutation $\{a_k | k \in N\}$, where N denotes the set of positive integers, we shall mean a sequence of positive integers such that every element of N appears once and only once in the sequence $\{a_k | k \in N\}$. Set $d_k = |a_{k+1} - a_k|$. The purpose of this paper is to answer, in the affirmative, two questions which were raised by Roger Entringer at the University of New Mexico.

Question 1. Can one construct a permutation $\{a_k | k \in N\}$ such that given any interger n, $|\{k | d_k = n\}| \leqq C$, where C is some fixed constant which is independent of n?

Question 2. Can one construct a permutation $\{a_k | k \in N\}$ such that $\{d_k | k \in N\}$ is also a permutation?

These questions are similar in nature to a problem described in [2] as having been solved by M. Hall. A solution by J. Browkin appears in [1], and the problem is to find a subset A of N such that every natural number is the difference of precisely one pair of numbers of the set A. Note that in this problem one considers all differences and not just differences formed by adjacent members in a sequence.

Let us consider the following procedure for constructing a sequence. Let $a_1 = 1$, $a_2 = 2$. We define a_3 as follows: Let a_3 be the smallest integer, which has not already appeared in the sequence, such that the difference $|a_3 - a_2|$ has also not appeared. Clearly, $a_3 = 4$. Assume that $a_1, a_2, \cdots, a_t$ have been defined in this way. Define a_{t+1} by the following conditions: (i) $|a_{t+1} - a_t| \neq d_i$, $i < t$, (ii) $a_{t+1} \neq a_i$, $i < t + 1$, and (iii) a_{t+1} is the smallest positive integer with properties (i) and (ii).

Clearly, every integer appears at most once in the sequences $\{a_k | k \in N\}$ and $\{d_k | k \in N\}$. But are these sequences permutations? The next theorem settles this question for the sequence $\{a_k | k \in N\}$.

THEOREM 1. *The sequence, $\{a_k | k \in N\}$, constructed above is a permutation.*