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# MULTIPLICATION ALTERATION AND RELATED RIGIDITY PROPERTIES OF ALGEBRAS

DAVE RIFFELMACHER

**Given an algebra $C$ over a commutative ring $k$ and an element (called a $C$-two-cocycle) $\sigma = \sum_i a_i \otimes b_i \otimes c_i$ in $C \otimes_k C \otimes_k C$ satisfying certain relations, Sweedler defined a new multiplication $*$ on $C$ by $x*y = \sum_i a_i x b_i y c_i$ for all $x, y$ in $C$ and denoted $C$ with this new multiplication by $C^\sigma$. This paper studies three rigidity properties which arise by asking whether:**

**(i) $C^\sigma \simeq C$ as algebras;**

**(ii) a certain functor from the category of $C$-bimodules to the category of $C^\sigma$-bimodules is an equivalence;**

**(iii) a certain functor from the category of algebras over $C$ to the category of algebras over $C^\sigma$ is an equivalence. For certain algebras over a field $k$ (including finite dimensional algebras possessing a Wedderburn factor), these rigidity properties are shown to be equivalent to (respectively): (i) all $k$-separable subalgebras $B$ of $C$ are commutative and for a separability idempotent $\sum_i x_i \otimes y_i$ of $B$, $\{c \in C \mid \sum_i x_i c y_i = 0\}$ is an ideal with square $\{0\}$; (ii) all $k$-separable subalgebras of $C$ are central; (iii) $k$ is the only $k$-separable subalgebra of $C$.**

We recall Sweedler's basic definitions [7] and determine some elementary properties of multiplication alteration in §§1 and 2. The behavior of an algebra under alteration by Waterhouse's $C$-two-cocycle $\sigma_e = e \otimes 1 + 1 \otimes e - (e \otimes 1)(1 \otimes e)$ associated with a $k$-separable subalgebra $B$ of $C$ having separability idempotent $e$ is studied in §3.

Section 4 introduces the notion of dominance: the $k$-algebra $C$ is said to dominate the $k$-algebra $D$ (written $C > D$) if there is a $C$-two-cocycle $\sigma$ with $D \simeq C^\sigma$. $C$ is called rigid if $C > D$ implies $D \simeq C$. Dominance is a partial order on the class of $k$-algebras. In the course of proving this an alternate characterization of a $C$-two-cocycle $\sigma$ in terms of the existence of a certain functor $F^\sigma: A(C) \rightarrow A(C^\sigma)$ is given. (For any $k$-algebra $D$, $A(D)$ is the category of $k$-algebras over $D$.) We provide a dominance description of the central simple algebras over a field $k$ as the "highly nonrigid" algebras and characterize those algebras over a perfect field $k$ with nilpotent Jacobson radical $J(C)$ and $k$-dim $C/J(C)$ finite which are rigid. The main step in our study of rigidity is a theorem which states that if the kernel of an idempotent algebra endomorphism $p$ of $C$ satisfies a certain nilpotency condition every $C$-two-cocycle $\sigma$ is "equivalent" to the $p(C)$-two cocycle $p(\sigma)$ (cf. Theorem 4.7).