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A COMPARISON OF THE RELATIVE UNIFORM TOPOLOGY AND THE NORM TOPOLOGY IN A NORMED RIESZ SPACE

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It is well-known that if a normed Riesz space (L, ρ) is ρ-complete, i.e., a Banach lattice, then the relative uniform topology and the norm topology are the same. Under weaker conditions the two topologies show some measure of agreement. In particular if L has (i) the local σ-property (a property weaker than both the σ-property and local completeness) and (ii) the property that for every ideal A the norm closure of A equals the set of limit points of relatively uniformly convergent sequences of elements of A, then every sequence $u_n \downarrow 0$ with $\rho(u_n) \to 0$ is a relatively uniformly convergent sequence. (This generalizes a theorem of Luxemburg and Zaanen.) However conditions (i) and (ii) are not sufficient to imply that the relative uniform topology and the norm topology agree on order intervals. Examples are given illustrating increasing degrees of agreement of the two topologies.

Let L be a Riesz space, $\{x_n: n \in N\}$ be a sequence in L, $x \in L$, and $v \in L^+$, then $\{x_n\}$ *converges v-uniformly* to x, written $x_n \to x$ (v-unif.), if there is a sequence $\{\alpha_n\}$ of real numbers such that $\alpha_n \downarrow 0$ and $|x_n - x| \leqq \alpha_n v$ for each n in N. A sequence $\{x_n\}$ of elements of L is said to *converge relatively uniformly* to $x \in L$, written $x_n \to x$ $(r.u.)$, if $x_n \to x$ (v-unif.) for some v in L^+. If S is a subset of L, define S'^{ru} to be the set of x in L such that there exists a sequence $\{x_n\} \subseteqq S$ with $x_n \to x$ $(r.u.)$. A subset S of L is said to be *relatively uniformly closed* if $S = S'^{ru}$, and the relatively uniformly closed sets are exactly the closed sets for a topology, the *relative uniform topology* τ_{ru}, on L. For an arbitrary subset S of L the set $\bar{S}^{\tau ru}$ is called the *relative uniform closure* of S and is denoted by $\bar{S}^{ru}$.

The relative uniform topology and its relation to the order structure on L have been investigated in some detail. (See [2].) We note here that although $S \subseteqq S'^{ru} \subseteqq \bar{S}^{ru}$ for every set S in L, it is not necessarily true that $S'^{ru} = \bar{S}^{ru}$. Recall that L is said to have the *σ-property* if every sequence of elements of L is contained in a principal ideal, i.e., given any sequence $\{u_n\}$ in L^+, there is a sequence $\{\lambda_n\}$ of positive numbers and an element $u \in L^+$ such that $u_n \leqq \lambda_n u$ for each $n \in N$. In particular if L has a strong unit, it has the σ-property.