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THE QUADRATIC AND QUARTIC CHARACTER OF CERTAIN QUADRATIC UNITS I

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Let ε_m denote the fundamental unit of the real quadratic field $Q(\sqrt{m})$. It is our purpose to evaluate the rational quadratic and biquadratic residue symbols of ε_m modulo a prime p for certain values of m.

We use the notation (ε_m/p) and $(\varepsilon_m/p)_4$ throughout this paper as rational quadratic and biquadratic residue symbols, interpreting $\sqrt{m}$ as an integer modulo p. In 1969 Barrucand and Cohn [1] proved, using the arithmetic of $Q(\sqrt{-1}, \sqrt{2})$, that if $p = 8n + 1$ is prime, so that $p = c^2 + 8d^2$, then

$$\left(\frac{\varepsilon_2}{p}\right) = \left(\frac{1+\sqrt{2}}{p}\right) = (-1)^d .$$

Since then a number of similar results have been obtained for certain other quadratic and quartic symbols using such tools as cyclotomy, rational biquadratic reciprocity laws, etc. (see Brandler [2], Lehmer [4], [5], [6]).

In this paper we apply the ideas of Barrucand and Cohn [1] in other biquadratic fields with unique factorization, thereby reproving some known results, proving some conjectures of E. Lehmer [6] and obtaining some additional new results. The method succeeds in the 21 imaginary bicyclic biquadratic fields having class number 1 and which contain $Q(\sqrt{-1})$, $Q(\sqrt{-2})$ or $Q(\sqrt{2})$ as a subfield. It would be interesting to know if similar techniques can be used in the remaining 26 imaginary bicyclic biquadratic fields with class number 1 or to determine octic symbols. (For a complete list of the imaginary bicyclic biquadratic fields with class number 1 see Brown and Parry [3].)

We now sketch the method used. First the quadratic or quartic symbol under consideration is expressed in terms of the representation of p by the indefinite form associated with the real quadratic subfield of the biquadratic field. This is accomplished using Jacobi's form of the law of quadratic reciprocity, and the results are given in the table below. In the case of those results involving quartic symbols it is first necessary to observe that $2\varepsilon_m$ is a square in the quadratic subfield, and this brings in the symbol $(2/p)_4$ whose value is well known, viz., if $p = 8n + 1$ is prime so that $p = a^2 + 16b^2 = c^2 + 8d^2$ then $(2/p)_4 = (-1)^b = (-1)^{n+d}$. Next we consider a prime