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SOME RESULTS ON PSEUDO-CONTRACTIVE MAPPINGS

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Let E be a Banach space and D a subset of E. A mapping $f: D \to E$ such that $||u - v|| \leqq ||(1 + r)(u - v) - r(f(u) - f(v))||$ for all $u, v \in D, r > 0$ is called pseudo-contractive. The basic result is the following: Let X be a bounded closed subset of E, suppose $f: X \to E$ is a continuous pseudo-contractive mapping such that $f[X]$ is bounded, and suppose there exists $z \in X$ such that $||z - f(z)|| < ||x - f(x)||$ for all $x \in$ boundary (X). Then $\inf\{||x - f(x)||: x \in X\} = 0$. If in addition X has the fixed point property with respect to nonexpansive self-mappings, then f has a fixed point in X. It follows from this result that if $T: E \to E$ is continuous and accretive with $||T(x)|| \to \infty$ as $||x|| \to \infty$, then $T[E]$ is dense in E, and if in addition it is assumed that the closed balls in E have the fixed-point property with respect to nonexpansive self-mappings, then $T[E] = E$. Also included are some theorems for continuous pseudo-contractive mappings f which involve demi-closedness of $I - f$ and consequently require uniform convexity of E.

1. Introduction. Let E be a Banach space, X a subset of E, and f a mapping of X into E. Then f is said to be *nonexpansive* if for all $x, y \in X$,

$$||f(x) - f(y)|| \leqq ||x - y||$$

while f is said to be *pseudo-contractive* if for all $x, y \in X$ and $r > 0$,

$$(1) \qquad ||x - y|| \leqq ||(1 + r)(x - y) - r(f(x) - f(y))|| .$$

The pseudo-contractive mappings (which are clearly more general than the nonexpansive mappings) derive their importance in nonlinear functional analysis via their firm connection with the accretive transformations: A mapping $f: X \to E$ is pseudo-contractive if and only if the mapping $T = I - f$ is *accretive*, i.e., for every $x, y \in X$ there exists $j \in J(x - y)$ such that

$$(2) \qquad \operatorname{Re}(T(x) - T(y), j) \geqq 0$$

where $J: E \to 2^{E^*}$ is the normalized duality mapping which is defined by

$$J(x) = \{j \in E^*: (x, j) = ||x||^2, ||j|| = ||x||\} .$$

(See Browder [3]; Kato [13].)