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THE LENGTH OF THE PERIOD OF THE SIMPLE CONTINUED FRACTION OF $d^{1/2}$.

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Let $p(d)$ denote the length of the period of the simple continued fraction for $d^{1/2}$ and ε the fundamental unit in the ring $Z\,[d^{1/2}]$. We prove that as $d \to \infty$,

THEOREM 1. $p(d) \leqq 7/2\pi^{-2}d^{1/2}\log d + O(d^{1/2})$.

THEOREM 2. $\log\varepsilon \leqq 3\pi^{-2}d^{1/2}\log d + O(d^{1/2})$.

THEOREM 3. $p(d) \neq o(d^{1/2}/\log\log d)$.

THEOREM 4. **If** $\log\varepsilon \neq o(d^{1/2}\log d)$ **then also**

$$p(d) \neq o(d^{1/2}\log d)\ .$$

Recently Hickerson [1] has proved that $p(d) = O(d^{1/2+\delta})$ for every $\delta > 0$, and in fact a result somewhat more precise than this. Lehmer [2] has suggested that for arbitrarily large d, $p(d)$ might be as large as $0.30d^{1/2}\log d$, and if this is indeed the case then Theorem 1 is almost the best possible result. In fact it is easy to show that $p(d) = O(d^{1/2}\log d)$ using known results regarding $\log\varepsilon$, but the constant in Theorem 1 improves the best obtainable in this way.

Let ε_0 denote the fundamental unit in the field $Q(d^{1/2})$, $[a_0, \overline{a_1, a_2, \cdots a_{p(d)-1}, 2a_0}]$ the continued fraction for $d^{1/2}$ and P_r/Q_r its rth convergent. Then as is well known $\varepsilon = \varepsilon_0$ or ε_0^3. Thus by the result of Stephens [3],

$$\log\varepsilon \leqq 3\log\varepsilon_0 \leqq \frac{3}{2}(1 - e^{-1/2} + \delta)d^{1/2}\log d\ .$$

Now $Q_0 = 1$, $Q_1 = a_1 \geqq 1$ and $Q_{r+2} = a_{r+2}Q_{r+1} + Q_r \geqq Q_{r+1} + Q_r$ and so by induction $Q_r \geqq u_{r+1}$, the Fibonacci number, for $r \geqq 0$. Now

$$\begin{aligned}\varepsilon &= P_{p(d)-1} + Q_{p(d)-1}d^{1/2}\\ &> 2d^{1/2}Q_{p(d)-1} - 1\\ &\geqq 2d^{1/2}u_{p(d)} - 1\\ &> \left\{\frac{1+\sqrt{5}}{2}\right\}^{p(d)},\end{aligned}$$

and so $p(d) < Ad^{1/2}\log d$ where A is approximately 5/4.

In exactly the same way, using $a_r < d^{1/2}$ for $0 \leqq r < p(d)$ it is possible to show that $p(d) \gg \log\varepsilon/\log d$. Since $d = 2^{2k+1}$ gives $\varepsilon = (1+\sqrt{2})^{2k}$, we find that for arbitrarily large d it is possible for $p(d) \gg d^{1/2}/\log d$, and it will be shown that this can be improved at