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MINIMAL AND MAXIMAL SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem

$$u'' = f(x, u, u'), \quad 0 < x < 1$$

$$B^i u = \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1$$

under rather mild assumptions on f. In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u'.

1. Introduction. This paper is concerned with the construction of the minimal and the maximal solutions of the nonlinear boundary value problem (BVP);

$$u'' = f(x, u, u'), \quad 0 < x < 1 \tag{1}$$

$$B^i u \equiv \alpha_i u(i) + \beta_i u'(i) = b_i, \quad i = 0, 1. \tag{2}$$

Obviously, when such boundary value problems are not necessarily uniquely solvable, the existence of the minimal and the maximal solutions plays a useful role in both the quantitative and qualitative theory for these classes of problems. Although considerable literature exists (see, for instance, [9]) about the min-max solutions of initial value problems, very little is known for boundary value problems even in the case of scalar equations (1)-(2). The results in the latter direction usually impose some kind of monotonicity assumption on f in its second and third arguments. In this paper, we establish the minimal and the maximal solutions of BVP (1)-(2) under rather mild assumptions on f. In particular, no assumption of monotonicity is made on $f(x, u, u')$ either in u or u'. The approach taken is essentially an extension of the ideas in [4] where a monotone method was developed for the quasilinear case when f depends on u' linearly. In this paper, we extend the results of [4] in two ways. First, we relax the restriction of linearity of f in u'. Secondly, while in [4] a linear iteration scheme was employed to generate a monotone sequence, here we require a nonlinear iteration scheme. This necessitates our proving existence and uniqueness of solutions of the nonlinear iteration scheme, whereas in the linear case one immediately has existence and uniqueness of the iterative procedure.