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OPERATORS INVERTIBLE MODULO THE WEAKLY COMPACT OPERATORS

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A continuous linear operator is a Fredholm operator if and only if it is invertible modulo the compact operators. In this note, we will generalize several theorems on Fredholm operators to theorems concerning operators invertible modulo the weakly compact operators.

1. **Preliminaries.** We fix the following notation.

$\boldsymbol{C}$ = the complex field

B = the category of complex Banach spaces and continuous linear operators

$B(X, Y)$ = the Banach space of continuous linear operators from X to Y (with the sup norm $|\ |$)

$WK(X, Y)$ = the closed subspace of all weakly compact operators in $B(X, Y)$

$X^* = B(X, \boldsymbol{C})$, the conjugate space

$F^* = B(F, \boldsymbol{C})$, the adjoint of $F: X \to Y$

I_X = the identity operator on X

$\bar{X} = X^{**}/n_X(X)$, where $n_X: X \to X^{**}$ is the natural injection.

If $F \in B(X, Y)$, then the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "n_X"] \arrow[d, "F"'] & X^{**} \arrow[r] \arrow[d, "F^{**}"'] & \bar{X} \arrow[r] \arrow[d, "\bar{F}"'] & 0 \\
0 \arrow[r] & Y \arrow[r, "n_Y"] & Y^{**} \arrow[r] & \bar{Y} \arrow[r] & 0
\end{tikzcd}$$

uniquely defines an operator $\bar{F} \in B(\bar{X}, \bar{Y})$. (Here n_X, n_Y are the natural injections.)

We will need the following results (1.1)-(1.7) from [9].

1.1. X is reflexive if and only if $\bar{X} = 0$. [9, (3.1)]

1.2. $F \in WK(X, Y)$ if and only if $\bar{F} = 0$. [9, (4.1)]

1.3. $\bar{I}_X = I_{\bar{X}}$. [9, (2.3)]

1.4. If $E \in B(X, Y)$ and $F \in B(Y, Z)$, then $\overline{FE} = \bar{F}\bar{E}$. [9, (2.3)]

1.5. $|\bar{F}| \leqq |F|$. [9, (2.3)]

1.6. For any $a, b \in \boldsymbol{C}$ and $E, F \in B(X, Y)$, $\overline{aE + bF} = a\bar{E} + b\bar{F}$. [9, (2.4)]

1.7. There exists a natural topological isomorphism $N_X: (\bar{X})^* \to (\overline{X^*})$. i.e., given any $F \in B(X, Y)$, the diagram