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FIXED POINT THEOREMS FOR MAPPINGS WITH A CONTRACTIVE ITERATE

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Several fixed point theorems are proved for metric-space mappings which satisfy a contractive condition involving an iterate of the mapping, where the iterate depends on the point in the space.

Let (X, d) denote a complete metric space. In [3] the second author has established fixed point theorems for mappings which satisfy a variety of contractive conditions. The common property of the mappings discussed in [3] is that the fixed point is unique, and can be found by using repeated iteration, beginning with some initial choice $x_0 \in X$.

The first result in this direction is that of V. M. Sehgal [5] who proved the following.

THEOREM S1. *Let (X, d) be a complete metric space, T a continuous self-mapping of X which satisfies the condition that there exists a real number k, $0 < k < 1$ such that, for each $x \in X$ there exists a positive integer $n(x)$ such that, for each $y \in X$,*

$$(1) \qquad d(T^{n(x)}(x)) , \quad T^{n(x)}(y) \leqq kd(x, y) .$$

Then T has a unique fixed point in X.

L. F. Guseman, Jr. [1], extended Sehgal's result by removing the condition of continuity of T and weakening (1) to hold on some subset B of X such that $T(B) \subset B$, where, for some $x_0 \in B$, B contains the closure of the iterates of x_0. Further extensions for a single mapping appear in [2] and in [4].

We shall be concerned with a pair of mappings which satisfy the following contractive condition.

Let T_1, T_2 be self-mappings of X such that there exists a constant k, $0 < k < 1$ such that there exist positive integers $n(x)$, $m(y)$ such that for each $x, y \in X$,

$$(2) \qquad \begin{aligned} d(T_1^{n(x)}(x) , \quad T_2^{m(y)}(y)) \leqq k \max \{ & d(x, y) , \quad d(x, T_1^{n(x)}(x)) , \\ & d(y, T_2^{m(y)}(y)) , \quad [d(x, T_2^{m(y)}(y)) + d(y, T_1^{n(x)}(x))]/2\} . \end{aligned}$$

THEOREM 1. *Let T_1, T_2 be self-mappings of a complete metric space (X, d) which satisfy (2). Then T_1 and T_2 have a unique common fixed point.*