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THE EXISTENCE OF DISCONTINUOUS MODULE DERIVATIONS

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In this paper it is shown that if a commutative Banach algebra B with identity has a maximal ideal M whose algebraic powers $M^2, M^3, \cdots$ form a descending chain of ideals which never becomes constant then there exists a discontinuous module derivation from B into a Banach-B-bimodule. This fact is linked with the known sufficient conditions for every module derivation from B to be continuous when B is separable. Some examples are given to demonstrate unusual behaviour in such chains of ideals in particular situations.

1. Introduction. Let B be a Banach algebra over the complex field and let X be a Banach-B-bimodule. We say that a linear mapping $D: B \rightarrow X$ is a module derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for a, b in B where * denotes the module operation. Recently some attention has been given to the problem of finding sufficient conditions on B so that every module derivation from B to any Banach-B-bimodule is continuous. Two sufficient conditions for the case when B is commutative and separable were described in [12] where it was noted that one of the conditions was also necessary for the automatic continuity of module derivations. In this paper we examine the extent to which the other condition is also necessary, proving that this is so under certain extra conditions on the maximal ideals of B. A related problem is to investigate the continuity of homomorphisms from B into some Banach algebra. This has been investigated in the general situation most recently by W. G. Bade and P. C. Curtis, Jr. [1]. Notice that the existence of a discontinuous module derivation from B into a Banach-B-bimodule implies the existence of a discontinuous homomorphism from B into some Banach algebra [20, p. 49]. We make no attempt to discuss the particular case, $B = C(\Omega)$, where Ω is an infinite compact Hausdorff space (all module derivations are continuous in this situation—see [12]), where the recent spectacular work of H. G. Dales [5] and J. Esterle [8] has shown that, assuming the continuum hypothesis, there exists a discontinuous homomorphism from $C(\Omega)$ into a Banach algebra. It follows from their work that, when B is an infinite dimensional commutative separable Banach algebra, there is a discontinuous homomorphism from B into some Banach algebra [6, 9] and it seems likely that the same result is true even when B is not separable.